# MONTHLY NOTICES

OF THE

# ROYAL ASTRONOMICAL SOCIETY

VOL. 92 1931 NOVEMBER 13 No. 1


Dr. H. KNOX-SHAW, PRESIDENT, in the Chair.

---

The election by the Council of the following Fellows was duly confirmed :—

Robert H. Baker, Ph.D., Professor of Astronomy and Director of the Observatory of the University of Illinois, Urbana, Ill., U.S.A. ;

Rev. John P. Delaney, S.J., A.B., A.M., Professor of Physics and Seismology, Canisius College, Buffalo, N.Y., U.S.A. ;

C. T. Elvey, Ph.D., Yerkes Observatory, Williams Bay, Wis., U.S.A. ;

Rudolph Moritz, K.C., B.Sc., 4 Brick Court, London, E.C. 4 ; and

Rev. Gilbert Lord Wilson, Ph.B., 342 N. Madison Avenue, Pasadena, Cal., U.S.A.

---

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Loring B. Andrews, Astronomical Laboratory, Harvard University, Cambridge, Mass., U.S.A. (proposed by H. Shapley) ;

Mrs. Violet Olivia Cressy-Marcks, Broad Walk, Buxton, Derbyshire, and 42 Park Lane, London, W. 1 (proposed by A. R. Hinks) ;

John Theodore Foxell, M.A., 12 Tavistock Road, Croydon, Surrey (proposed by A. E. Levin) ;

H. Groot, 52 Huizerweg, Bussum, Holland (proposed by E. A. Kreiken) ;

Miss Mira Francisca Hardcastle, Stephen's Orchard, Grayshott, Hindhead, Surrey (proposed by Francisca Herschel) ;

Charles Henry Kebby, 52 Lumley Road, Horley, Surrey (proposed by S. Chapman) ;

Alfred Knight, 595 Madison Avenue, New York City, N.Y., U.S.A. (proposed by B. R. Baumgardt) ;

Ernest Vivian Leembruggen, 31 Gordon Street, Suva, Fiji (proposed by G. T. McCaw) ;

Arumugam Wisuvalingam Mailvaganam, B.A., B.Sc., Emmanuel College, Cambridge (proposed by W. M. Smart) ;

Edward B. Maitland, O.B.E., B.Sc., A.M.I.Mech.E., Craig House, Woodlands Road, Surbiton (proposed by T. E. R. Phillips) ;

Mrs. Marion Field Skinner, 17 Solle-shott West, Letchworth, Herts (proposed by Mary Proctor) ;

Michael Heriot Huth Walters, B.A., King's College, Cambridge (proposed by W. M. Smart) ;

Reginald Hugh Wickramasinghe, B.A., B.Sc., c/o The Colonial Secretary, Ceylon (proposed by W. M. Smart) ;

Charles Llewelyn Carrick Williams, Beckets, Rusper, Sussex (proposed by E. A. Reeves) ; and

Kenneth Young, D.S.O., 40 Danethorpe Road, Wembley, Middlesex (proposed by L. J. Comrie).

---

Two hundred and thirty-four presents were announced as having been received since the last meeting, including :—

Astrographic Catalogue : Catania Observatory, vol. 7, pte. 3, 4 ; vol. 8, pte. 3, 4 ; Tacubaya Observatory, vol. 1, pte. 2 ;

Astrographic Chart : San Fernando Observatory, zone $-3°$ (10 charts) ;

Bulletin Astronomique, Mémoires et Variétés, Série 2, Tomes 2–6 ; Revue générale, Tomes 2–5 and appendice (presented by Paris Observatory) ;

Royal Observatory, Greenwich, Photographic Magnitudes of Stars brighter than $14^{m}{\cdot}0$ in Kapteyn's Selected Areas ;

Nautical Almanac, Appendices, 1834–1854 (presented by Dr. L. J. Comrie) ;

Lorentz (H. A.), Problems of Modern Physics (presented by Dr. R. A. Millikan) ;

Newton (Isaac), Naturvetenskapens Matematiska Principer, Boken II, III, Svensk översättning av C. V. L. Charlier (presented by Professor C. V. L. Charlier).

---

The President announced that the late Mr. Adolph Lindemann (a Fellow of the Society since 1873) had bequeathed to the Society the sum of one thousand pounds, free of legacy duty.

## *The Recession of the Extra-galactic Nebulæ.*

By Sir A. S. Eddington, F.R.S.

A systematic motion of recession of the spiral nebulæ was found by V. M. Slipher in his determinations of radial velocity. E. P. Hubble showed further that the speed of recession is approximately proportional to the distance, amounting to about 500 km. per sec. per megaparsec distance. This distribution of motion represents a uniform inflation of the system of the galaxies, extending at least as far as the observations extend.

An inflation of this kind (or, alternatively, an equally systematic contraction) is predicted by relativity theory. The theoretical deduction has become much more forceful since it was first put forward as a possibility by W. de Sitter. The expansion depends on the cosmical constant $\lambda$ in the law of gravitation $G_{\mu\nu} = \lambda g_{\mu\nu}$ for empty space. At first the $\lambda$ term was regarded dubiously as a fancy addition ; but now that we have a better understanding of the significance and origin of the law, it has become evident that $\lambda$ cannot be zero. Further, although the matter of the universe might be adjusted so as to form an equilibrium configuration (Einstein universe) which does not display this phenomenon, it is now known that the equilibrium is unstable, so that sooner or later the phenomenon must occur. The modern definitive form of the theory is due to A. Friedman and G. Lemaître.

But although theory predicts the phenomenon qualitatively, it has hitherto given no hint as to its scale. So far as relativity theory is concerned we do not know whether the effect will be perceptible within range of our telescopes ; the recession might be appreciable at nebular distances, or it might require distances $10^6$ or $10^{60}$ times greater. This is a weak point in the claim that the observed recession of the nebulæ is actually the theoretically predicted expansion. The observed recession, if accepted as genuine, is alarmingly rapid ; the universe must have doubled its radius within geological times, and the consequences in regard to the time-scale and the problem of stellar evolution are most startling. For this reason there has been some disinclination to admit the genuineness of the velocities of the nebulæ, and it is suggested that the red-shift of their light has some other interpretation ; those who take this view need not dispute the theoretical prediction of an expanding universe, since the theory does not assert that the expansion will be rapid enough for us to detect.

In a recent investigation * I think I have shown that the value of $\lambda$ and the consequent rate of expansion of the universe can be calculated by pure theory. I find that distant objects will recede with a velocity

528 km. per sec. per megaparsec.

---

* *Proc. Roy. Soc.*, A, **133**, 605, 1931.

No astronomical data are used in obtaining this result ; the constants in the formulæ are the ordinary physical constants measured in the laboratory. This result agrees closely with the observed recession of the nebulæ.

Assuming that my theory is correct, it follows that the observed recession is actually the de Sitter-Lemaître effect. The system of the galaxies is doubling its dimensions every 1300 million years, and the longest time-scale of evolution becomes altogether incongruous. It seems unlikely that the age of our galaxy can much exceed $10^{10}$ years.

This agreement of theory and observation supplies a much needed check on the accuracy of the accepted scale of astronomical distances. When we count up the number of intermediate steps in comparing the distance of a nebula with the standard metre, and remember the risky extrapolations involved in some of the stages, it is gratifying that the scale should be confirmed to within, say, 20 per cent.

The result, 528 km. per sec. per mp., is the full cosmical expansion, neglecting the countervailing gravitational attraction of the nebulæ on one another. The correction for gravitational attraction is probably not large, though it will not be entirely negligible. The following table is calculated from the formulæ of Lemaître's theory. The first column gives the decimal logarithm of an assumed mean density (gm. per cu. cm.) of matter in the universe ; the second column gives the recession (km. per sec. per mp.) corrected for the corresponding gravitational attraction ; the third column gives the corresponding expansion factor, *i.e.* the ratio of the present radius of the universe to the radius in the equilibrium Einstein state from which it presumably started to expand.

| $\text{Log}_{10}\rho$ | Recession | Expansion Factor |
|---|---|---|
| $-\infty$ | 527·8 | $\infty$ |
| −30 | 520·1 | 9·83 |
| −29·5 | 511·7 | 6·70 |
| −29 | 494·3 | 4·56 |
| −28·5 | 459·0 | 3·11 |
| −28 | 388·6 | 2·17 |

Determinations of the mean density within our galaxy around the Sun give about $10^{-23\cdot5}$. Allowing for the comparatively empty regions between the galaxies, there does not seem much likelihood of a world-density so high as $10^{-28\cdot5}$ ; but it is difficult to give any limits with confidence. According to the table we should expect to find an actual velocity of recession between 460 and 530, probably nearer to the latter figure.

In the following outline of the theory I omit any detailed justification of the successive steps, since full particulars will be found in the paper quoted.

Considering any linear constant of nature such as the radius of the normal hydrogen atom, what do we mean by saying that it is a *constant length?* There is no such thing as absolute length ; so we can only mean that its ratio to some other length is constant. In practice this other length is the

standard metre ; but the complicated system of molecules forming the standard rod is only introduced as an intermediary, and has nothing to do with our ideal conception of the constancy of a " constant of nature." I have shown * that, according to the law of gravitation, what we call a metre at any place and in any direction is always a constant fraction of the radius of curvature of space-time associated with that place and direction ; this is simply a translation of the equation $G_{\mu\nu} = \lambda g_{\mu\nu}$ from symbols into words. Thus, dispensing with the irrelevant intermediary, we can answer the question proposed : the constancy of the radius of the hydrogen atom means that it is always the same fraction of the radius of curvature of the world at the place and in the direction that it occupies.

Now consider the physical equation which we use to determine the radius of the hydrogen atom. It is obvious that it cannot give any result at all unless it brings in both the hydrogen atom and the system with which its dimensions are being compared ; and when the equation is in its simplest fundamental form, the comparison length is a radius of curvature of space-time. The idea of my theory is to find the equation of wave-mechanics in its simplest and most fundamental form, expressing directly the relation of the linear quantities characteristic of the hydrogen atom (or other elementary systems) to the radius of curvature of space-time. The ordinary form drags in the standard metre like a red herring, and makes it difficult to discern what the equation is really trying to express.

Consider a spherical space of radius $R$ in which there is a mobile electron at a distance $r$ from a fixed electron (or proton). If we draw a picture we shall mark two fixed points, viz. the fixed electron and the centre of spherical space, and call $r$ the radius vector of the mobile electron from the first and $R$ the radius vector from the second. In fact $r$ and $R$ are bipolar co-ordinates. Thus the whole problem may be considered to be based on a " core " in which $r$ and $R$ occur symmetrically.

Now take the wave equation as ordinarily accepted in quantum theory and look for this core. We immediately see that the term containing $r$ is proportional to $1/r$. For definiteness we shall suppose the whole equation divided through by the appropriate numerical factor so that this term is simply $1/r$. Evidently there must be a term $1/R$ to complete the core symmetrically.

I have no doubt that the correct wave equation for the problem stated would be found to contain the core

$$\frac{1}{r} + \frac{1}{R}.$$

But the problem in its literal sense is not a practical one, since it concerns a universe containing only two electrons. Now the practical wave equation (based on experiment) relates to a universe containing, say, $N$ electrons. All the $N$ electrons have the radius vector $R$ and only one (an unspecified one of the $N$ electrons) has the radius vector $r$. Without entering into details

* *Mathematical Theory of Relativity*, § 66. See also *Nature of the Physical World*, chap. vii.

the general principles of the quantum theory appear to indicate that this will make the core become

$$\frac{1}{r}+\frac{\sqrt{N}}{R}\,.$$

It may be mentioned that if a wave function $\psi$ represents one electron, $\psi\sqrt{N}$ represents $N$ electrons ; so that when the above core is multiplied by a wave function (as in the wave equation) it becomes

$$\frac{1}{r}\psi+\frac{1}{R}\psi\sqrt{N}=\frac{1}{r}\psi(1)+\frac{1}{R}\psi(N).$$

I conclude that a term $\sqrt{N}/R$ should appear in the wave equation (when it is reduced to the form in which the term due to a fixed electron is $1/r$). There are a number of terms in the practical wave equation, but only one of them is independent of the position of the mobile electron ; this term is a natural constant $mc^2/e^2$. Clearly this is the term which must be identified with $\sqrt{N}/R$. Our theory therefore gives the result

$$\frac{\sqrt{N}}{R}=\frac{mc^2}{e^2}, \tag{1}$$

$N$ being the number of electrons in the universe and $R$ the radius of the universe in the Einstein configuration.*

$e^2/mc^2$ is the constant $b=2{\cdot}81\,.\,10^{-13}$ cm. used in researches on the stellar absorption coefficient † and regarded as a sort of conventional radius of the electron. Our result can therefore be written

$$R=b\sqrt{N}. \tag{2}$$

We are now able to find $R$ and $N$ separately because another equation relating them is given by the ordinary relativity theory of the Einstein world. Assuming that the number of protons is equal to the number of electrons, the mass of the universe is approximately $N_M$, where $M$ is the mass of a proton. The condition for an Einstein world is

$$\frac{GN_M}{c^2}=\tfrac{1}{2}\pi R. \tag{3}$$

From (1) and (3) the following values are obtained :—

$R=1{\cdot}010\,.\,10^{27}$ cm. $=328$ megaparsecs $=1068$ million light-years
$M=2{\cdot}143\,.\,10^{55}$ gm. $=1{\cdot}08\,.\,10^{22}\times\odot$
$N=1{\cdot}29\,.\,10^{79}$
$\rho_e=1{\cdot}05\,.\,10^{-27}$ gm. cm.$^{-3}$,

$M$ being the mass of the universe and $\rho_e$ the density of the Einstein configuration. The speed of recession of distant objects (uncorrected for gravitational attraction) per unit distance is $c/R\sqrt{3}$, which, with the above

* I omit here the part of the theory which shows that $R$ is the Einstein radius and not the instantaneous radius of the expanding universe.

† *Internal Constitution of the Stars*, p. 395.

value of $R$, gives 527·8 km. per sec. per mp. The foregoing numerical values differ very little from those given in my paper, " The Instability of Einstein's Spherical World," * which were based on an adopted observational value $c/R\sqrt{3} = 500$.

When gravitational attraction becomes negligible the distances of the galaxies double themselves in a time $(R\sqrt{3}/c) \log_e 2$, or

1280 million years.

This makes their apparent magnitudes grow fainter at the constant rate

$1^{m}\cdot18$ per $10^9$ years.

(At extremely great distances corrections for curvature and for Doppler effect will be required.)

Another interesting formula obtainable from (1) and (3) is

$$N = (\tfrac{1}{2}\pi F)^2, \tag{4}$$

where $F$ is the ratio of the electrostatic to the gravitational force between a proton and electron according to the classical formulæ.

---

## *The Expanding Universe.*

By W. H. McCrea, Ph.D., and G. C. McVittie, Ph.D.

1. *Introduction.*—In this note we discuss an oversight which occurred in some previous work dealing with the effect of the formation of mass-particles in an Einstein universe on the total volume of the universe. We also venture to criticise some related work by the Abbé Lemaître, and we conclude by pointing out the relation of these investigations to a recent suggestion of Einstein.

The precise interpretation we attach to the problem of the " expansion " or " contraction " of the universe from an equilibrium state, due to redistribution of the matter in it, is this : In the universe in equilibrium, an observer is supposed to measure the total volume of space and the total proper mass of the matter contained in it. He can then express this volume in terms of the mass and of the velocity of light. The matter is now supposed to be redistributed and the universe constrained to take up the corresponding equilibrium configuration. The observer can again find an expression for the volume in terms of the new measured mass and the velocity of light (measured at points distant from singularities). If, then, the velocity of light and the total proper mass are assumed to remain constant, he can say how the volume has changed. But unless this, or some similar, assumption is made the problem would appear meaningless.

The particular case of this general problem which we considered was that

* *M.N.*, **90**, 668, 1930.

of departure from the equilibrium state known as the " Einstein universe." The line-element of this universe can be put in either of the forms :

$$ds^2 = dt^2 - \frac{dq^2}{1 - q^2/R^2} - q^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1)$$

or

$$ds^2 = dt^2 - \frac{1}{(1 + r^2/4R^2)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \qquad (1a)$$

In a first paper,* basing ourselves on (1), we found that, if a condensation of mass $m$ formed at the origin, the metric was given rigorously by

$$ds^2 = \left(1 - \frac{2m}{q}\right)dt^2 - \frac{dq^2}{\left(1 - \frac{2m}{q}\right)\left(1 - \frac{(q-m)^2}{R^2}\right)} - q^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2)$$

In a subsequent paper,† one of us, starting from the form (1$a$), gave a general method for treating the case of $n$ particles distributed at random in the universe, when the masses of the particles were supposed small. For the present purpose it is sufficient to take the special case of this result when $n = 1$ and the particle is at the origin. Then the metric is

$$ds^2 = \left\{1 - m\left(\frac{1}{r} - \frac{r}{4R^2}\right)\right\}dt^2 - \frac{1 + m\left(\frac{1}{r} - \frac{r}{4R^2}\right)}{(1 + r^2/4R^2)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \qquad (3)$$

Here the square and higher powers of $m$ are neglected.

We then found (paper I) that the volume of the space given by (2) was less than that of the Einstein universe (1), whilst (paper II) that of (3) was greater. The difference was attributed to the special conditions imposed in paper I. The investigations were carried out to the first approximation only, *i.e.* neglecting powers of $m$ higher than the first and taking $m/R$ small compared with unity.

We have now to point out that both the metrics (2) and (3) possess a singularity at infinity, a circumstance which was unfortunately overlooked in our previous work. When this is properly allowed for, both metrics agree in showing that the presence of particles *does not affect the volume of space to the first order.* This result may be expected to hold generally. It means that the presence of a condensation may or may not determine an expansion or contraction of the universe, but that a method such as ours, depending on first-order perturbations, cannot decide the question.

2. *The Singularity at Infinity.*—Consider first the case of (3). This evidently has a singularity at $r = \infty$ . To see its nature, change the co-ordinate $r$ to $r'$ by means of

$$rr' = 4R^2. \qquad (4)$$

* W. H. McCrea and G. C. McVittie, *M.N.*, **91**, 128, 1930. *Referred to as I.*

† G. C. McVittie, *M.N.*, **91**, 274, 1931. *Referred to as II.*

Then (3) becomes

$$ds^2 = \left\{1 + m\left(\frac{1}{r'} - \frac{r'}{4R^2}\right)\right\}dt^2 - \frac{1 - m\left(\frac{1}{r'} - \frac{r'}{4R^2}\right)}{(1 + r'^2/4R^2)^2}(dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2)). \quad (3')$$

This is precisely of the same form as (3), with $r'$ replacing $r$, and $-m$ replacing $+m$. Hence, if the singularity at $r = 0$ corresponds to a particle of mass $+m$, as its derivation was intended to ensure, then the singularity at $r' = 0$ *corresponds to a particle of mass* $-m$. Physically, we look upon this as meaning that mass $m$ has been transferred from $r = \infty$ to $r = 0$. Similarly, if we take the form (20) of paper II for $n$ particles, there will correspond to each particle a *negative* particle at its antipodes.

With regard to the metric (2), the presence of the second singularity is obscured by the use of the co-ordinate $q$. We recall that the Einstein universe of radius $R$ is usually represented, in the first instance, by

$$ds^2 = dt^2 - R^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\} \quad (5)$$

with

$$0 \leqslant \chi \leqslant \pi\,; \quad 0 \leqslant \theta \leqslant \pi\,; \quad 0 \leqslant \phi \leqslant 2\pi. \quad (6)$$

(1) is obtained from (5) by putting

$$q = R\sin\chi, \quad (7)$$

whilst (1*a*) is got by putting

$$r = 2R\tan\frac{\chi}{2}. \quad (8)$$

From (7) it is apparent that $q$ is not suitable for representing the whole range $0 \leqslant \chi \leqslant \pi$, since to each value of $q$ correspond two values of $\chi$. If the whole universe is to correspond to this range of $\chi$, and not to $0 < \chi \leqslant \frac{\pi}{2}$, a co-ordinate such as that given by (8) must be employed. In (2), therefore, transform from $q$ to $r$ by means of

$$q = R\sin\chi + m \quad \text{and} \quad r = 2R\tan\frac{\chi}{2} \quad (9)$$

and we obtain, to the first order in $m$,

$$ds^2 = \left\{1 - \frac{2m}{r}\left(1 + \frac{r^2}{4R^2}\right)\right\}dt^2 - \frac{\left\{1 + \frac{2m}{r}\left(1 + \frac{r^2}{4R^2}\right)\right\}}{(1 + r^2/4R^2)^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} \quad (2')$$

where $r$ now ranges from 0 to $\infty$.

If now we change to $r'$ by (4), it is clear that the form (2′) is unaltered. Hence, if the singularity at $r = 0$ corresponds to a mass $+m$, *then that at* $r' = 0$ *also corresponds to one of mass* $+m$.

3. *The Null Result.*—Proceeding now to the calculation of the total volume of (3) in terms of the proper mass, we see that paper II (32) becomes

$$M^{\times} = (1 - \tfrac{5}{4}\eta)\frac{\pi}{2}R,$$

since each particle now has an equal and opposite particle at its antipodes so that the total mass of the condensations in the universe is zero. Using this value of $M^{\times}$ in the subsequent work, it is easily seen that $\eta = 0$ and the volume is

$$V = \frac{16}{\pi} M^{\times 3}.$$

To this order of approximation, therefore, the volume remains the same as for the original Einstein universe.

Using the metric (2), we have now to add a further mass $m$ to that given in paper I (20) and we obtain

$$M^{\times} = \frac{\pi}{2} R\left(1 + \frac{4}{\pi}\frac{m}{R}\right).$$

Combining this with I (23) we again have the result

$$V = \frac{16}{\pi} M^{\times 3}.$$

showing no change in volume to our order of approximation.

This null result would appear to be capable of generalization, since it holds for any number of particles of the kind treated in Paper II. This would mean that the equilibrium volume of the universe possesses a stationary value corresponding to the Einstein state—uniform density and zero pressure. It renders much more difficult the problem we set out to solve; for it means that a higher approximation than the first is necessary, and this at present seems unattainable owing to the difficulty of ensuring that the condition of isotropy of pressure, the expression for the element of proper mass, and certain other necessary corrections, shall all hold to the higher order desired.

4. *Lemaître's Treatment of the Problem.*—The Abbé Lemaître has recently * put forward a different treatment of the question, and concluded that the formation of condensations has nothing to do with the disturbance of the equilibrium of the universe. This he ascribes to a phenomenon he terms " stagnation." It seems, however, that his work is open to criticism on various grounds.

His method is to suppose a condensation to be the centre of a gravitational field with radial symmetry. He then postulates the existence of a " neutral zone " in this field at which the gravitational effect of the condensation is balanced by the average gravitational effect of the rest of the matter in the universe. The study of the apparent motion of the neutral zone will then give information about the apparent expansion or contraction of space.

He studies this motion first by finding the equation of motion of a material particle placed in this zone. But he apparently imposes no condition which ensures that the particle is actually in this zone and not at a general point of the field. Hence his method merely gives a general relation between the velocity and acceleration of the particle and the pressure, $p$, of the matter at the point occupied by it. Further, he arbitrarily lays down the condition

* G. Lemaître, *M.N.*, **91**, 490, 1931.

$p+\rho=0$ where $\rho$ is the density of the matter in the universe. He assigns no physical reason for this and it seems to destroy the applicability of the result to real phenomena, except in the case when both $p$ and $\rho$ are zero. In that case, however, he is dealing with a condensation which has formed in a de Sitter universe. This is an altogether different problem from the one we considered. It involves the introduction of matter into an empty universe, not the redistribution of matter already present in an equilibrium state.

As an alternative treatment he supposes a thin material shell placed at the neutral zone and studies its motion. It is difficult to see how anything is gained by this, for each particle of the shell must behave exactly like the material particle used in his first method. We suggest that had the condition for a neutral zone actually been introduced in the first method, and the condition $p+\rho=0$ dispensed with, the result of the second method would have been obtained. At any rate the result now obtained is

$$\frac{2}{R}\frac{d^2R}{ds^2}+\frac{1}{R^2}\left(\frac{dR}{ds}\right)^2+\frac{1}{R^2}=\lambda-\kappa P, \qquad (10)$$

where $R$ is a constant multiple of the radius, $r$, of the neutral zone, and $P$ is the proper pressure there. From the fact that this is of the same *form* as the equation of the pressure in an expanding or contracting universe and contains no *explicit* mention of the condensation, he infers that the formation of the condensation does not affect the rate of expansion or contraction.

But this is a long and uncertain step. We object to it on the ground that, until $P$ (which Lemaître does not evaluate) is known at the neutral zone, nothing can be said about the motion of the latter. Further, our work tends to show that, even to the first approximation, $P$ *does* depend on the mass of the condensation. Until Lemaître can show that $P$, at the neutral zone, does not depend upon it, his conclusion is not justified. But even so, it seems clear that the complete solution of equation (10) would give merely the relation between the radius of the neutral zone (proportional to $R$) and the distance ($s$) along the geodesic traversed by the particle. Until $s$ is given in terms of the proper time of some observer we do not see that the equation (10) gives any information as to the expansion or contraction of the universe.

Lemaître then proceeds to a further study of homogeneous universes, *i.e.* those without condensations. He arrives at the result that a suitable change in $p$ will start an expansion. This appears to be equivalent to a result already given by Eddington.* What we wish to discover is some agency which will *make* $p$ vary.

In his concluding section Lemaître takes a homogeneous universe in equilibrium, with pressure $p=p_0$, radius $R=R_1$. He then supposes the pressure suddenly reduced to zero, his "stagnation" process, while the radius remains equal to $R_1$. The new state $p=0$, $R=R_1$ is therefore not an equilibrium state. He then proceeds to discuss the *subsequent* motion. All his results consequently apply to a universe made arbitrarily to start from

* Sir A. S. Eddington, *M.N.*, **90**, 668, 1930.

this non-equilibrium state. These are admittedly the results he requires for his estimates of how the rate of expansion depends on the magnitude of the initial " stagnation." At the same time the investigation does not explain how the universe contrives initially to pass from the equilibrium state $p = p_0$, $R = R_1$, to the non-equilibrium state $p = 0$, $R = R_1$. It tells us what will happen *after* the equilibrium has been disturbed in a particular way, but provides no mechanism which will *produce* such a disturbance.

We conclude by stressing again the two conditions which, in our view, must be observed in an approach to this subject (*cf.* § 1 above):

(*a*) The initial state must be some equilibrium state (in our case an Einstein universe).

(*b*) There must be some " condition of permanency," such as the constancy of total proper mass, to ensure that we are dealing throughout all changes with the same universe.

5. *Einstein's Universe.*—One reason for retaining the $\lambda$-term in the gravitational field equations was that without it they gave a negative pressure for a positive uniform density of matter. Einstein * has lately pointed out that this conclusion holds only if the field is restricted to be static. If this restriction is removed he shows that there can be a positive uniform density with zero pressure for zero value of $\lambda$. He proposes, therefore, to drop the $\lambda$-term from the field equations.

The point that concerns us here is that, if we take his resulting equation for the radius of space, $R$, then $d^2R/dt^2$ is never zero, *i.e.* there is no equilibrium state. On this view our original problem ceases to exist. We have no longer to find an agency for disturbing an equilibrium state, since no such state can exist.

It would be out of place, however, to attempt here a full examination of Einstein's proposal. Our general conclusion is that our original problem is still unsolved, but that this new suggestion may destroy any physical interest in solving it.

---

*The Observations of Eros, 1930–31. Notes on Methods of Reduction and Publication of Results.* By H. Spencer Jones, F.R.S., Chairman, Solar Parallax Commission, I.A.U.

1. The following notes on methods of reduction of the observations of Eros secured around the time of the opposition of 1931 February 17, and on the form of publication, have been prepared in order to secure uniformity in the published data with a view to facilitating the subsequent co-ordination of results from different observatories, and to draw attention to a few points of importance which may be overlooked. At the recent opposition, the photographic observations preponderate largely over visual observations ; these notes are therefore concerned mainly with the reduction of photographic observations.

* A. Einstein, *Sitzb. d. Preuss. Akad. d. Wiss.*, **12**, 235, 1931.

2. *Corrections for Differential Refraction.*—In observations obtained at large zenith distances, it is necessary to apply corrections for the second order terms in differential refraction. It should be noted that such terms change sign with the parallax, and therefore enter with full effect into the derivation of the solar parallax from the combination of observations secured at large eastern and western hour angles.

The correct formulæ to use in the reduction of photographic plates are

$$\Delta x = -k\{X(2+X^2)x^2 + Y(1+2X^2)xy + X(1+Y^2)y^2\},$$
$$\Delta y = -k\{Y(1+X^2)x^2 + X(1+2Y^2)xy + Y(2+Y^2)y^2\},$$

where

$x$, $y$ denote the measured co-ordinates of a star ;

$X$, $Y$ denote the co-ordinates of the zenith on the plate.

$X$, $Y$ are given by

$$X = \sin\omega \tan\zeta, \qquad Y = \cos\omega \tan\zeta,$$

$\omega$ being the parallactic angle and $\zeta$ the zenith distance. In the above formulæ, $\Delta x$, $\Delta y$ are corrections applicable to the measured co-ordinates to obtain the co-ordinates free from the effects of the second-order terms. These corrections should be applied before making the solutions for the ordinary six constant reductions.

The application of these squared terms is facilitated by the diagrams given by Hinks.*

It should be noted that Kapteyn † and Baillaud ‡ derived expressions for the second-order terms in differential refraction in rectangular co-ordinates, which are equivalent to one another, but differ from the above expressions. These formulæ are incorrect, although the expressions for differential refraction in distance and position angle, from which they were deduced, are correct. The error in the formulæ given by Kapteyn and Baillaud arises as follows : if $S_1$, $S_2$ denote the true positions of two stars, $S_1$ being at the centre of the plate, and the arc $S_1S_2$ is denoted by $\sigma$, the projection of $S_1S_2$ on the plate is $\tan\sigma$ ; if $S_1'$, $S_2'$ are the positions of $S_1$, $S_2$ as affected by refraction and if $S_1'S_2' = \sigma + \alpha\sigma$, it is assumed by Kapteyn and Baillaud that the projection of $S_1'S_2'$ on the plate is $\tan(\sigma + \alpha\sigma)$. This is not correct to the second order.

Attention is drawn to this, because incorrect formulæ were used in the reduction of some of the observations obtained at the 1901 opposition.

The principal refraction terms depending upon the change of the constant of refraction with zenith distance and upon the square of the constant of refraction are linear, and are automatically absorbed into the orientation and scale constants.

3. *Mean of Several Exposures.*—In general, each plate contains more than one exposure on Eros. It is important to examine within what limits it is legitimate to assume that the parallax and position corresponding

* *M.N.*, **63**, 138–147, 1903 January.

† *Bulletin du Com. Int. Perm. de la Carte du Ciel*, **3**, 54, 1902.

‡ *Ibid.*, **3**, 19, 1902 (*vide* p. 48).

to the mean time of several exposures can be assumed to be equal respectively to the mean values of the parallax and position for the several exposures.

The limits for the parallax are very narrow. The parallax in right ascension is

$$\pi_\alpha = \pi_0 \cos\phi \sec\delta \sin h = \pi \sin h \text{ (say)},$$

where $\pi_0$ is the horizontal parallax and $h$ is the hour angle. Suppose the hour angle at the mean time of several exposures is $h$, and $t_1, t_2 \ldots t_n$ are the mid-times of several exposures, measured from the mean time, so that $\Sigma t = 0$.

The parallax at the mean time is $\pi \sin h$.

The mean of the parallax values for the several exposures is

$$\pi \sin h - \tfrac{1}{2}\pi \sin h \,.\, \frac{\Sigma t^2}{n}.$$

The parallax at the mean time is thus greater than the mean of the true values by

$$\delta\pi_\alpha = \tfrac{1}{2}\pi \sin \mathrm{h} \,.\, \frac{\Sigma t^2}{n}$$

$$= \tfrac{1}{2}\pi_\alpha \cdot \frac{\Sigma t^2}{n}.$$

If the time is expressed in minutes,

$$\frac{\delta\pi_\alpha}{\pi_\alpha} = \frac{1}{2n} \cdot \left(\frac{\pi}{720}\right)^2 \Sigma t^2.$$

The deduced value of the parallax will be systematically too small. If there are four exposures on the plate, each of $3^m$ duration, with intervals of $1^m$ between the exposures, $\delta\pi_\alpha/\pi_\alpha = \cdot 0015/8$. The deduced value of the solar parallax will be too small by $0''\cdot 0016$. If there are three exposures of $3^m$ duration, with intervals of $1^m$, the deduced value will be too small by $0''\cdot 0009$. These quantities are not negligible compared with the probable errors to be expected from the programme. Therefore it is necessary in the derivation of the solar parallax from observations at east and west hour angles to reduce each exposure separately.

In the same way, in the case of observations taken near the meridian, it is not in general legitimate to assume that the parallax in declination corresponding to the mean time of several exposures is equal to the mean parallax for the several exposures.

If a single exposure extends over a sufficient time, the mean parallax for the exposure will not be equal to the parallax at the mid-time of the exposure. But as the exposures on Eros were for the most part short, no error is in general to be feared on this account. In any case of doubt it is easy to ascertain what error is involved.

The limits within which it may be assumed that the position corresponding to the mean time of several exposures is equal to the mean position

for the several exposures are considerably wider than in the case of the parallax. The maximum second difference for intervals of one day is about $5^s$ in right ascension and $115''$ in declination. If the time-interval is less than about 30 minutes, the mean position may be assumed equal to the position at the mean time. But since this is not the case for the parallactic displacement, it is generally necessary to reduce each exposure independently.

4. *Measurement and Reduction of Plates.*—The reduction of the measurements on the plates follows the methods customary for astrographic plates, for which various formulæ are available. As some of the co-operating observatories are not familiar with astrographic reductions, suitable formulæ are summarised below.

If $A$, $D$ are the right ascension and declination of the centre of the plate ; $\alpha$, $\delta$ the right ascension and declination of a star whose rectangular co-ordinates are $\xi$, $\eta$, we have :

$$\begin{aligned}\xi &= \sin(\alpha - A)\sec(\theta - D)\sin\theta\cot\delta \\ &= \tan(\alpha - A)\sec(\theta - \mathrm{D})\cos\theta, \\ \eta &= \tan(\theta - D),\end{aligned}$$

where

$$\tan\theta = \sec(\alpha - \mathrm{A})\tan\delta.$$

Thus

$$\theta = \delta + \tan^2\frac{\alpha - A}{2}\sin 2\delta \text{ (approximately)}$$

gives $\theta$, and $\xi$, $\eta$ are determined from the above formulæ when $A$ and $D$ are known. The standard co-ordinates of the reference stars can thus be computed. The measured co-ordinates, corrected for the second-order terms in differential refraction, are equated to the computed co-ordinates by linear formulæ, and the constants of these formulæ derived. The standard co-ordinates of Eros can thus be derived.

Knowing $\xi$, $\eta$, we can derive $\alpha$, $\delta$ by the formulæ

$$\tan(\alpha - A) = \frac{\xi\sec D}{1 - \eta\tan D},$$

$$\delta = D + \eta - \tfrac{1}{2}\xi^2\tan(D + \eta) - \frac{\eta^3}{3} + \text{fourth power and higher terms.}$$

Or it is possible to use the special tables given by Loewy in Circular 10 of the Astrographic Conference. These tables were computed primarily for the reduction of the Eros observations at the opposition of 1901. They extend to declination $55^\circ\ 20'$ and therefore may be used also for the reduction of the present series of observations.

With these tables, $x_0$, $y_0$ denote the measured co-ordinates relative to the centre of the plate (corrected for orientation and scale). Then

$$x = x_0 - \alpha x_0 \quad \text{(Table I gives } \alpha x_0\text{, arg. } x_0, y_0\text{)};$$
$$y = y_0 - \alpha y_0 \quad \text{(Table II gives } \alpha y_0\text{, arg. } y_0\text{)}.$$

If $A$, $D$ denote the right ascension and declination of the centre of the plate, $\alpha$, $\delta$ the right ascension and declination of a star :—

$$(\alpha - A)\cos\delta = x + t_a \quad \text{(Table III gives } t_a\text{, arg. } x_0 \text{ and } \delta\text{)};$$
$$D + y = \delta + t_a \quad \text{(Table IV gives } t_a\text{, arg. } D + y \text{ and } x\text{)}.$$

$t_a$ changes sign with change in the sign of $x$.
$t_a$ is independent of the sign of $x$.
$t_a$ is unaltered with change in sign of $\delta$.
$t_a$ changes sign with $\delta$ ; *i.e.* for negative declinations, $t_a$ is to be treated as negative.

Loewy's tables are not ideally arranged but their use will save a considerable amount of computing.

In view of the fact that the observations will be used for the dual purpose of deriving the solar parallax and also the mass of the Moon, it is desirable that all photographic plates obtained should be measured both in right ascension and in declination. For the plates taken at large east and west hour angles, the measures in declination are of little value for the derivation of the solar parallax ; similarly, for the co-operative programmes between northern and southern observatories, the measures in right ascension are of little value for the derivation of the solar parallax. The measures in both co-ordinates will serve, however, to give a value of the mass of the Moon. The maximum value of the lunar equation was about $2^s{\cdot}25$ in right ascension and $23''$ in declination, so that the declination measures will contribute appreciably to the weight of the derived value of the mass of the Moon.

5. *Comparison Stars.*—The primary reference stars, contained in the three lists published by Professor Kopff, will serve as the comparison stars in the measurement of plates of astrographic or shorter focal length. In order to eliminate as far as possible systematic errors arising from differential atmospheric dispersion, it is advisable that stars of extreme colour or spectral type (say, types B–A0, K5 and M) should be excluded. The spectral types of these stars have been determined at Harvard, and their colour indices have been determined both by Seares, Sitterley and Miss Joyner * at Mount Wilson and by Ross and Zug † at Yerkes. There is very little information available as to the colour of the secondary comparison stars, though it appears probable that the mean spectral type of these stars is not likely to differ very much from the colour of Eros.

For the measurement of photographic plates which cover a field of less than $2^\circ \times 2^\circ$, comparison stars should be selected as far as possible from the list of secondary reference stars which has been published by Professor Schorr.‡ Observers are recommended to select from this list suitable reference stars for each plate obtained, before measurement of the plates is commenced, and to prepare a list of the secondary reference stars for which positions will be required. The path of Eros has been photographed at the

* *Ap. J.*, **72**, 311, 1930.
† *Ast. Nach.*, **239**, 289, 1930.
‡ *Ibid.*, **239**, 321, 1930 ; **240**, 65, 1930 ; **242**, 249, 1931.

Bergedorf, Greenwich, Leipzig, Lick and Cape Observatories. If each co-operating observatory will forward, with as little delay as possible, to each of these observatories lists of secondary comparison stars for which positions are required, the work of measurement of the path plates will be lightened, as comparison stars for which positions are not required by any co-operating observatory need not be measured.

In the case of micrometric observations, the star or stars nearest to Eros had in general to be used. Many such stars are not contained in Professor Schorr's list of secondary comparison stars. Micrometric observers are requested to forward lists of comparison stars for which positions are required to the observatories which have obtained series of path plates ; where such stars are not contained in Professor Schorr's list, the approximate right ascension and declination should be given.

The final reductions of the photographic plates obtained with long-focus instruments and of the micrometric observations will not be possible until the various measures of the path plates have been completed and collated.

6. *Form of Publication of Results.*—The primary reference star places, as given in the catalogue prepared by Professor Kopff, form the basis to which the positions both of secondary reference stars and of Eros itself will be referred. The star places in the primary reference star catalogue are given for the mean equinox of 1930·0.

The definitive ephemeris of Eros, with which the observed places of Eros will be compared, will also be referred to the mean equinox of 1930·0 and will be barycentric, *i.e.* referred to the centre of gravity of the Earth-Moon system. The lunar equation will be given separately.

The position of the planet, derived from the measurements on the photographic plates, is thus referred directly to the mean equator and equinox of 1930·0. Precession and nutation do not enter into consideration, being allowed for automatically on the reduction. The circular aberration terms are also automatically removed from the place of the planet in the process of reduction. The aberration terms depending upon the eccentricity of the Earth's orbit are not removed, however, since the mean places of the stars contain these terms as usual.

If, therefore, we add the circular aberration terms to the place of the planet, derived from the photographs, the resulting place will contain the whole aberration due to the motion of the Earth's centre. The correction for parallax is also applied to reduce to geocentric place.

The whole aberration (circular + eccentricity terms) is equivalent to the alteration in the place of the planet during the light-time. The usual process of antedating the time of observation by the light-time will thus give the true geocentric position corresponding to the antedated time. It should be noted that the reduction to geocentric place (parallax correction) should be computed for the actual time of observation and not for the antedated time. The diurnal aberration terms are automatically removed in the reduction of the plate, being common to stars and planet.

The final ephemeris with which the observations are compared will be barycentric, as stated above. The observed place of the planet is therefore

to be corrected first to geocentric place (correction for diurnal parallax) and then to barycentric place (correction for lunar equation computed for the actual time of observation, not for the antedated time), and the circular aberration terms are to be added. The position so obtained is to be compared with the antedated position of the planet derived from the barycentric ephemeris. Proceeding in this way, the small aberration term produced by the motion of the Earth about the centre of gravity of the Earth and Moon (lunar aberration) is eliminated. If we antedate in a geocentric ephemeris, we allow both for the aberration due to the motion of the Earth's centre and also for the lunar aberration. We are therefore allowing for too much, as the lunar aberration is automatically eliminated in the reduction of the plate.

For the computation of the diurnal parallax, the basic value, 8″·800, of the solar parallax should in all cases be used.

It is desired that, in the publication of results, both the G.M.T. (or Universal Time or *Weltzeit*) and the local sidereal time of the observations should be given. Errors not infrequently enter into the reduction from one to the other ; if both are given, there is a check on such errors as well as on typographical errors. The time-interval or intervals of the observation period or periods corresponding to each derived position of the planet should also be given, in order that it can be tested whether it is permissible to use the parallactic factor for the mean time or whether terms of the second order must be taken into account.

If the reductions are carried through to the final stage of deriving the $\Delta\alpha$, $\Delta\delta$ differences (observed minus ephemeris), the following data should be given for each derived position : G.M.T. and local S.T. of observation ; time-interval of observation ; observed position of planet ; circular aberration ; parallax ; lunar equation ; deduced barycentric place ; ephemeris position (for antedated time) ; observed *minus* ephemeris positions. If preferred, to avoid the delay which may result if the publication is withheld until the definitive ephemeris becomes available, the computation of the parallax, lunar equation and ephemeris position can be omitted.

It is desirable that details of the method of observation should be stated, *e.g.* whether Eros or a star was guided upon, in one or both co-ordinates ; whether or not the magnitude of Eros was reduced to equality with that of the comparison stars ; whether the method of interrupted exposures was used ; etc.

In the case of micrometric observations, if preliminary publication is decided upon, it will be best if the observer gives merely the values of $\Delta\alpha$ and $\Delta\delta$, reduced to 1930·0, with an identification of the comparison star used for each observation. For these data, approximate star places suffice.

## *Observations of Nova Pictoris.*

By W. H. van den Bos and W. S. Finsen.

A few early observations of the duplicity of Nova Pictoris were published in *M.N.*, **88**, 488, 1928. The present paper embodies all our later observations. We are indebted to Dr. J. Voûte for permission to include here his measures made with the 23½-inch refractor at Lembang, Java ; they are in excellent agreement with our results, when allowance is made for the difficulty of the object.

| Date | Hour Angle | Position Angle | Distance | Definition * | Power | Difference of Mag. | Observer † | |
|---|---|---|---|---|---|---|---|---|
| | h | ° | ″ | | | m | | |
| 1928·243 | +0·9 | 72·0 | 0·48 | 2 | 410 | 0·0 | B | |
| ·243 | +1·0 | 66·0 | ... | 2 | 530 | ... | B | |
| ·243 | +0·4 | 66·3 | 0·54 | 3+ | 410 | 0·2 | φ | Combined magnitude estimated 7·7. Twilight ; nuclei separated. Diameter of nebulosity : major axis 2″·14, minor axis 1″·25. |
| ·243 | +0·6 | 64·8 | 0·48 | 3 | 810 | 0·1 | φ | Nuclei not easily seen with this power and definition. |
| ·246 | +0·8 | 64·5 | 0·45 | 2+ | 410 | 0·0 | B | |
| ·246 | +0·6 | 70·2 | 0·48 | 2+ | 410 | 0·0 | φ | Definition too bad for 810 ×. Combined magnitude estimated 7·5. |
| ·249 | +0·7 | 72·5 | 0·52 | 2 | 410 | 0·0 | B | |
| ·249 | +1·0 | 73·2 | 0·53 | 2 | 410 | ... | B | Red field. |
| ·249 | +0·7 | 69·0 | 0·53 | 2 | 410 | 0·0 | φ | Distinct minimum between nuclei in good moments. Combined magnitude estimated 7·4. |
| ·249 | +1·0 | 80·2 | 0·43 | 2 | 410 | ... | φ | In bright red field, which makes image look less nebulous, and minimum brightness in centre more distinct. |
| ·251 | +0·9 | 72·0 | 0·47 | 3–5 | 410 | −0·1 | φ | Red field. Definition variable ; in occasional good moments nuclei seem beautifully sharp and separated. *Sp* component seems certainly the fainter. Seems also to be closer than on previous nights. |
| ·251 | +1·2 | 70·7 | 0·34 | 2–4 | 810 | ... | φ | Red field. Seems no doubt distance is smaller than previously measured. |

* Scale : 5 = very good, 1 = very bad.
† B = van den Bos, V = Voûte, φ = Finsen.

| Date | Hour Angle | Position Angle | Distance | Definition * | Power | Difference of Mag. | Observer † | |
|---|---|---|---|---|---|---|---|---|
| | h | ° | ″ | | | m | | |
| 1928·254 | +0·8 | 68·1 | 0·44 | 3 | 810 | 0·0 | B | Red field. Suspect more than double. |
| ·254 | +1·3 | 76·0 | 0·40 | 3 | 410 | ... | B | Red field. |
| ·254 | +1·1 | 69·6 | 0·34 | 3+ | 810 | 0·0 | φ | Red field. Clearly separated. In moments of good definition strongly suspect a third nucleus *Nf*. |
| ·254 | +1·4 | 84·5 | 0·43 | 3 | 410 | ... | φ | Red field. Wire on S. side of nebula; position angle of *Sp* nucleus ? |
| ·254 | +1·4 | 66·7 | ... | ... | ... | ... | φ | Wire on N. side of nebula; position angle of *f* nucleus ? |
| ·257 | +1·1 | 64·0 | 0·49 | 3+ | 530 | ... | B | Red field. Does not look simply double. |
| ·257 | +1·3 | 74·2 | 0·46 | 3+ | 810 | 0·4 | B | Red field. |
| ·257 | +1·4 | 76·4 | 0·38 | 3 | 810 | 0·3 | φ | Red field. Third nucleus suspected *Sf*, 150° ±. |
| ·260 | +1·4 | ... | ... | 3 | 810 | ... | B | *AB* 99° ±; *AC* 248° ±; *AD* 37° ±. Some doubt. |
| ·260 | ... | ... | ... | 3 | 810 | ... | φ | *AB* 131° ± (almost certain); *AC* 260° ±; *AD* 2° ± (very doubtful, but consistently suspected). |
| ·276 | +1·4 | ... | ... | 2 | 810 | ... | B | Red wires. *AB* 87°·3, 0″·47; *AC* 243°·7. Doubt. |
| ·279 | +1·5 | ... | ... | 3 | 810 | ... | B | *A*, *B*, *C*, *D* glimpsed. |
| ·279 | +1·6 | ... | ... | 3 | 810 | ... | φ | At first glance image looks granular, and watching enables glimpses of each nucleus (*A*, *B*, *C*, *D*) to be caught. Impossible to measure as all four not seen simultaneously. |
| ·284 | +1·4 | ... | ... | 2+ | 810 | ... | φ | *AB* 124° ±; *AC* 253° ±; *AD* 353° ±. *B* and *C* regarded as certain; *D* somewhat doubtful. Later with superb definition all four distinctly and simultaneously seen for a few seconds, through a K2 filter. |

The existence of the components *A*, *B* and *C* was at this stage regarded as certain, while *D* was strongly suspected. The interpretation of the earlier measures of duplicity presents some difficulties. It is possible that the nova was simply double at that time and that the triple or quadruple character developed later. An alternative explanation is that the observers were able to see more detail as they grew more accustomed to the densely nebulous image.

* † See footnotes on p. 19.

| Date | Hour Angle | Position Angle | Distance | Definition * | Power | Difference of Mag. | Observer † | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | *AB* | | |
| | h | ° | ″ | | | m | | |
| 1928·301 | +2·3 | (111) | ... | 2 | 810 | 0·5 | B | Certainly seen earlier, but definition went off before measure. |
| ·301 | +2·1 | 144·9 | 0·29 | 3 | 810 | 0·3 | φ | |
| ·334 | +2·3 | 85·0 | 0·4 est. | 3 | 810 | 0·5 | B | Certain. |
| ·334 | ... | 105 ± | ... | ... | 810 | ... | φ | |
| ·339 | +2·3 | 92·5 | 0·35 | 2 | 810 | 0·7 | B | Unreliable. |
| ·358 | +3·4 | 101·2 | 0·4 est. | 2+ | 810 | 0·6 | B | Guess. |
| ·358 | +3·6 | 103·1 | 0·28 | 2 | 530 | 0·7 | φ | |
| | | | | | | *AC* | | |
| 1928·301 | +2·2 | 251·1 | 0·37 | 3 | 810 | 1·0 | B | Certain. |
| ·301 | +2·0 | 256·8 | 0·37 | 3+ | 810 | 0·5 | φ | |
| ·334 | +2·3 | 235·0 | 0·4 est. | 3 | 810 | 1·0 | B | Certain. |
| ·334 | ... | 251 ± | ... | ... | 810 | ... | φ | |
| ·339 | +2·2 | 244·1 | 0·39 | 2+ | 810 | 1·0 | B | Unreliable. |
| ·358 | +3·3 | 231·6 | 0·4 est. | 2+ | 810 | 1·0 | B | Guess. |
| ·358 | +3·7 | 235·2 | 0·39 | 2 | 530 | 1·5 | φ | |
| | | | | | | *AD* | | |
| 1928·301 | +2·2 | 346·3 | 0·28 | 2+ | 810 | 1·0 | φ | Difficult. |
| ·358 | +3·8 | 6·0 | 0·2 est. | 2 | 530 | 2·0 | φ | Guess. |

| | Range of Position Angle | Mean Distance |
|---|---|---|
| *AB* | (85° – 145°) | 0″·35 ± |
| *AC* | (231° – 257°) | 0″·38 ± |
| *AD* | 356° ? | 0″·25 ? |

It is evident that the apparent directions of the companions were still greatly affected by the nebulous and asymmetrical appearance of the central nucleus in this opposition.

* † See footnotes on p. 19.

*Opposition* 1928–29

| Date | Hour Angle | Position Angle | Distance | Definition * | Power | Difference of Mag. | Observer † | |
|---|---|---|---|---|---|---|---|---|
| | | | | *AB* | | | | |
| | h | ° | ″ | | | m | | |
| 1928·974 | −2·6 | 70·2 | ... | ... | 410 | 1·0 | B | |
| ·974 | −2·3 | 73·9 | 0·49 | 3 | 530 | ... | B | |
| ·974 | −2·5 | 63·9 | ... | ... | {410 / 530} | 1·0 | φ | |
| ·974 | −2·1 | 51·1 | 0·42 | ... | 530 | ... | φ | |
| 1929·041 | +0·7 | 66·6 | 0·78 | 4 | 810 | 2·0 | B | Splendid image, nuclei look stellar. |
| ·041 | +0·6 | 67·0 | 0·33 | 4+ | 810 | 2·0 | φ | Nuclei sharp and stellar, although involved in dense nebulosity. *B* on edge of nebulosity. |
| ·065 | ... | 67·1 | 0·59 | ... | ... | 2·0 | φ | Measure unreliable ; through haze. |
| ·076 | +1·1 | 65·3 | 0·62 | 3 | 810 | 2·5 | B | |
| ·181 | +0·5 | 65·0 | 0·63 | 3+ | 530 | 1·5 | B | |
| ·181 | +0·4 | 59·5 | 0·55 | 4+ | 530 | 0·8 | φ | |
| | | | | *AC* | | | | |
| 1928·974 | −2·6 | 230·0 | ... | 3 | 410 | 1·5 | B | |
| ·974 | −2·3 | 235·7 | 0·54 | 3 | 530 | ... | B | |
| ·974 | −2·5 | 220·5 | ... | ... | {410 / 530} | 2·0 | φ | |
| ·974 | −2·1 | 213·9 | 0·51 | ... | 530 | ... | φ | |
| 1929·041 | +0·7 | 236·5 | 0·76 | 4 | 810 | 2·5 | B | Splendid. |
| ·041 | +0·6 | 242·6 | 0·40 | 4+ | 810 | 2·5 | φ | Nuclei sharp and stellar, although involved in dense nebulosity. *C* perhaps just clear of nebulosity. |
| ·065 | ... | 220·7 | 0·5 est. | ... | ... | 3·0 | φ | Measure unreliable ; through haze. |
| ·076 | +1·1 | 222·3 | 0·68 | 3 | 810 | 2·0 | B | |
| ·181 | +0·5 | 234·8 | 0·76 | 3+ | 530 | 2·0 | B | |
| ·181 | +0·4 | 228·6 | 0·64 | 4+ | 530 | 1·5 | φ | |
| | | | | *AD* | | | | |
| 1929·041 | +0·7 | 6 | 0·3 est. | 4 | 810 | 3·0 | B | Strongly suspected. |
| ·041 | +0·6 | 14 est. | 0·25 est. | 4+ | 810 | 3·0 | φ | Suspected. |
| ·181 | +0·4 | 4 est. | 0·25 est. | 4+ | 530 | ... | φ | Suspected. |

| Means : | | Position Angle | Distance | | |
|---|---|---|---|---|---|
| | *AB* | 1929·068 | 65°·0 | 0″·55 | $\Delta m$ 1$^m$·6 | 8*n* |
| | *AC* | 1929·068 | 229°·4 | 0″·60 | $\Delta m$ 2$^m$·1 | 8*n* |
| | *AD* | 1929·088 | 8° ? | 0″·27 est. ? | $\Delta m$ 3$^m$ ? | 3*n* |

The general appearance of the nova seemed unaltered, although the companions and the nebulosity seemed decidedly fainter. The distances

* † See footnotes on p 19.

of *B* and *C* had undoubtedly increased, *C* being on the very edge of the nebula or perhaps just clear. The nuclei were always noted as sharp or stellar, and were more easily measured than in the preceding opposition. The apparent change in the position angle of *AB* is probably not real, but due to errors of measurement in the earlier part of 1928. *D* remained doubtful.

*Opposition* 1929–30

| Date | Hour Angle | Position Angle | Distance | Definition * | Power | Difference of Mag. | Observer † | |
|---|---|---|---|---|---|---|---|---|
| | | | | *AB* | | | | |
| | h | ° | ″ | | | m | | |
| 1929·758 | −1·5 | 73·6 | 0·83 | 5 | 600 | ... | V | |
| ·775 | −0·5 | 73·6 | 0·88 | 5 | 860 | ... | V | |
| ·791 | −0·3 | 72·6 | 0·85 | 4 | 860 | ... | V | |
| 1930·071 | −2·0 | 54 est. | 0·4 est. | 2 | 410 | 1·0 | φ | Clouds or thick haze. |
| ·082 | +1·1 | 70·2 | 0·75 | 3 | 440 | 2·0 | B | All look nebulous. |
| ·098 | +0·4 | 72·7 | 0·90 | 4 | 440 | 2·0 | φ | *A*, *B* and *C* sharper than last year. |
| ·101 | +0·5 | 77·8 | 0·86 | 5 | 440 | 2·0 | φ | Measured with ease; nuclei sharp and stellar. Distance undoubtedly greatly increased. |
| ·147 | +0·5 | 74·9 | 0·86 | 2+ | 440 | 2·0 | B | |
| ·164 | −0·2 | 76·6 | 0·79 | 5 | 860 | ... | V | |
| | | | | *AC* | | | | |
| 1929·758 | −1·5 | 228·1 | 0·66 | 5 | 600 | ... | V | |
| ·775 | −0·5 | 229·4 | 0·70 | 5 | 860 | ... | V | |
| ·791 | −0·3 | 227·3 | 0·76 | 4 | 860 | ... | V | |
| 1930·071 | −2·0 | 219 est. | 0·65 est. | 3 | 410 | 3·0 | φ | Clouds or thick haze. *C* somewhat fainter and more difficult. |
| ·082 | +1·1 | 228·0 | 0·74 | 3 | 440 | 2·5 | B | All look nebulous. |
| ·098 | +0·4 | 241·3 | 1·02 | 4 | 440 | 2·5 | φ | *A*, *B* and *C* sharper than last year. |
| ·101 | +0·5 | 232·8 | 1·02 | 5 | 440 | 2·5 | φ | Measured with ease; nuclei sharp and stellar. Distance undoubtedly greatly increased. |
| ·147 | +0·5 | 231·9 | 0·90 | 2+ | 440 | 2·5 | B | |
| ·164 | −0·2 | 221·2 | 0·62 | 5 | 860 | ... | V | |
| | | | | *AD* | | | | |
| 1930·082 | +1·1 | 20 est. | ... | 3 | 440 | ... | B | Suspected only. |
| ·098 | +0·4 | 10 est. | 0·3 est. | 4 | 440 | ... | φ | Suspected. |
| ·101 | +0·5 | ... | ... | 5 | 440 | ... | φ | Suspected. |

| Means : | | Position Angle | Distance | | |
|---|---|---|---|---|---|
| *AB* | 1929·990 | 74°·0 | 0″·84 | $\Delta m$ 1$^m$·8 | 8*n* |
| *AC* | 1929·990 | 230°·0 | 0″·80 | $\Delta m$ 2$^m$·6 | 8*n* |
| *AD* | 1930·090 | 15° ? | 0″·3 ? | ... | 2*n* |

* † See footnotes on p. 19.

The observations during this opposition showed that the steady increase in the distances *AB* and *AC* had been maintained. The nebulosity was decidedly fainter, while the nuclei retained their sharp appearance. The decrease in brightness was most apparent in the case of *C*, which was sometimes rather difficult. The presence of *D* was suspected.

Voûte remarks that his measures were made without any difficulty.

*Opposition* 1930–31

| Date | Hour Angle | Position Angle | Distance | Definition * | Power | Difference of Mag. | Observer † | |
|---|---|---|---|---|---|---|---|---|
| | | | | *AB* | | | | |
| | h | ° | ″ | | | m | | |
| 1930·974 | +1·0 | 57·7 | 0·87 | 1 + | 440 | 2·0 | φ | |
| 1931·005 | +0·8 | 68·0 | 0·83 | 2 | 440 | 2·5 | φ | |
| ·057 | −0·5 | 77·1 | 1·14 | 2 | 440 | 3·0 | B | Poor, very nebulous. |
| ·057 | −0·4 | 68·3 | 0·76 | 1 | 440 | 3·0 | φ | Poor. |
| ·101 | −0·4 | 65·7 | 1·02 | 4 | 440 | 9·0, 11·0 | B | Hard to measure. |
| ·101 | −0·6 | 71·2 | 0·86 | 4 | 440 | 8·5, 10·5 | φ | |
| ·134 | −0·6 | 82·1 | 1·21 | 4 | 440 | 9·0, 12·5 | B | |
| ·134 | −0·4 | 83·6 | 0·92 | 4 | 440 | 9·0, 12·0 | φ | All sharp. |
| | | | | *AC* | | | | |
| 1930·974 | +1·0 | 223·0 | 1·00 | 1 + | 440 | 3·0 | φ | *C* difficult now. |
| 1931·005 | +0·8 | 234·5 | 0·86 | 2 | 440 | 3·5 | φ | |
| ·057 | −0·1 | 238·0 | 0·90 | 2 | 440 | 3·5 | B | Poor, very nebulous. |
| ·057 | −0·3 | 228·8 | 1·15 | 1 + | 440 | 3·5 | φ | Poor, but seen without doubt. |
| ·101 | −0·4 | 224·2 | 1·18 | 4 | 440 | 9·0, 12·5 | B | Most difficult. |
| ·101 | −0·5 | 217·9 | 0·86 | 4 + | 440 | 8·5, 12·0 | φ | *C* sharp point, clear of nebulosity. |
| ·134 | −0·5 | 241·6 | 0·98 | 4 | 440 | 9·0, 13·0 | B | |
| ·134 | −0·4 | 219·9 | 0·91 | 4 | 440 | 9·0, 13·0 | φ | All sharp. |

| Means : | | Position Angle | Distance | | |
|---|---|---|---|---|---|
| *AB* | 1931·070 | 59°·2 | 0″·95 | 8*n* | mags. 8·9, 11·5 |
| *AC* | 1931·070 | 226°·0 | 0″·98 | 8*n* | mags. 8·9, 12·5 |
| *D* | Not seen. | | | | |

The observations made during this opposition were nearly all attended by great difficulty owing to the increasing faintness of the companions. *C* was especially difficult, and it seems unlikely that good measures will be possible in future unless the nebulosity becomes much fainter. There has been no perceptible increase in the size of the nebulosity since its discovery in 1928.

From the complete series of observations it may safely be concluded

* † See footnotes on p. 19.

that the distances of both $B$ and $C$ from $A$ have certainly increased and that both have faded more rapidly than the central star or nucleus, while there is no reliable evidence of change in the directions.

Though Nova Pictoris is the only example of its class where the formation and development of companions has been witnessed, it may be well to remember that it is probably not the only one where such a phenomenon has taken place. There seem to be strong reasons for believing in a similar occurrence in the case of $\eta$ Argus, the two near companions of that star having been discovered by Innes in 1914 and 1915. The chief star of $\eta$ Argus shows a decidedly nebulous disc, which cannot be focussed in the $26\frac{1}{2}$-inch refractor. The companions *may* be nebulous, but are too faint for a definite statement.

$\eta$ Argus is frequently classed as an irregular variable or pseudo-nova, as its behaviour has been abnormal—if the term " normal behaviour " has any meaning for novæ. Its light-curve is, however, a typical nova-curve, except that the time-scale is unusually large. It seems doubtful whether this fact and the peculiar character of its spectrum are sufficiently strong reasons to prevent its being classed with the novæ so long as no completely satisfactory theory of the formation of novæ has been given. It may be significant that Nova Pictoris has likewise been termed a " slow " nova, though its development has been much more rapid than that of $\eta$ Argus.

---

*On Astronomical Refraction.* By H. C. Plummer, F.R.S.

1. The difficulties and complexities of the treatment of astronomical refraction on the classical lines laid down by Bessel are well known. They are not made more attractive by the fact that the structure of the atmosphere, so far as it is understood at all, is certainly different from what it was formerly supposed to be. Hence any attempt to simplify the theory may be justified.

The curious (and inconvenient) thing is that, when the apparent zenith distance $\zeta$ is small, the law of refraction is expressed so well by $A \tan \zeta$. For this law, and any form of analytical development, must stop short at a certain point. The refraction, so far from becoming infinite, is always an essentially small angle. Even at the horizon it is only 0·01 when expressed in circular measure.

An expression of the law of mean refraction accurately by a simple function over the whole range from the zenith to the horizon can scarcely be expected; but it is easy to represent its general run in this way. Thus for the refraction in minutes the function

$$R' = \tan (\zeta^\circ - \zeta'), \qquad (1)$$

where the argument is meant to convey $59\zeta/60$ in a more easily remembered form, may serve this purpose. As an approximate formula it may be considered poor; but it agrees with the truth at $\zeta = 0$ naturally, and also at $\zeta = 50^\circ$ and about $\zeta = 89^\circ{\cdot}5$. Between 0 and $50^\circ$ the formula is in excess to

the maximum extent of 0″·3, and afterwards in growing defect to the extent of 1″ at 61° and nearly 2′ at 88° ; but at the end of the range this tendency is sharply over-corrected, so that the horizontal refraction appears as 38′·2. At the lowest altitudes the formula (1), without a single constant to be remembered, seems to represent the refraction about as well as it can be represented by a single homogeneous (Cassini) shell.* One difference there is between the formula and the refraction : $\tan(1-m)\zeta$ increases all the way to $\zeta = 90°$, while the refraction ceases to increase as the horizon is closely approached, though this property could hardly be detected by observation.

2. The familiar integral which represents the refraction, namely,

$$R = \int_1^{\mu_0} \frac{\mu_0 a \sin\zeta}{(\mu^2 r^2 - \mu_0^2 a^2 \sin^2\zeta)^{\frac{1}{2}}} \cdot \frac{d\mu}{\mu}, \tag{2}$$

can be written in the form

$$R = \mu_0 a \tan\zeta \int_1^{\mu_0} \frac{d\mu}{\mu^2 r}\left\{1 + \left(1 - \frac{\mu_0^2 a^2}{\mu^2 r^2}\right)\tan^2\zeta\right\}^{-\frac{1}{2}}$$

$$= \mu_0 a \tan\zeta \int_1^{\mu_0} \frac{d\mu}{\mu^2 r}\left\{1 - \tfrac{1}{2}\left(1 - \frac{\mu_0^2 a^2}{\mu^2 r^2}\right)\tan^2\zeta + \cdots\right\}$$

so long as $\tan\zeta$ is sufficiently small. Hence at small zenith distances

$$R = A\tan\zeta - B\tan^3\zeta \tag{3}$$

approximately, where

$$A = \mu_0 a \int_1^{\mu_0} \frac{d\mu}{\mu^2 r}; \tag{4}$$

$$B = \tfrac{1}{2}\mu_0 a \int_1^{\mu_0} \frac{d\mu}{\mu^2 r}\left(1 - \frac{\mu_0^2 a^2}{\mu^2 r^2}\right). \tag{5}$$

It is well known that with $A = 58''\cdot294$, $B = 0''\cdot0668$, constants determined directly from observations, the formula (3) represents the mean refractions tolerably as far as $\zeta = 75°$. Within this range no hypothesis regarding the atmospheric structure is therefore necessary.

3. It can be inferred that the factor $(1 - \mu_0^2 a^2/\mu^2 r^2)$ in the integral $B$ is effectively small. In a very rough sense its mean value is $2B/A = 1/436$. Now the horizontal refraction is

$$C = \mu_0 a \int_1^{\mu_0} \frac{d\mu}{\mu^2 r}\left(1 - \frac{\mu_0^2 a^2}{\mu^2 r^2}\right)^{-\frac{1}{2}}. \tag{6}$$

By using the mean value for the radical in the integrand it can be estimated that the magnitude of $C$ is of the same order as $21A$ or 20′. This value is within a half of the truth, and no better result can be expected from such perfunctory reasoning. So far as it goes it is confirmatory evidence that $r/a$ differs little from 1, and that the upper atmosphere plays a very small part in astronomical refraction. Otherwise stated, $C^2 = A^3/2B$ exactly when $\mu r$ is constant, an unpromising hypothesis if only for the reason that it implies an atmosphere 1·2 miles high.

* *Cf.* Turner, *M.N.*, **68**, 503, 1908.

4. When (2) is written in the form

$$R=\mu_0 a \sin\zeta\int_b^{\mu_0 a}\frac{d(\mu r)}{\mu r(\mu^2r^2-\mu_0{}^2a^2\sin^2\zeta)^{\frac{1}{2}}}\frac{rd\mu}{d(\mu r)}, \tag{7}$$

where $b$ is the value of $r$ at which $\mu=1$, it is clear that the integration requires the factor $rd\mu/d(\mu r)$ to be expressed as a function of $\mu r$. The simplest hypothesis is to suppose that this factor is a constant, say $-1/n$, since it is obviously negative. Then

$$\frac{\mu r}{\mu}\cdot\frac{d\mu}{d(\mu r)}+\frac{1}{n}=0;$$

$$\mu^{n+1}r=\text{const.}=\mu_0{}^{n+1}a=b \tag{8}$$

which expresses the Bouguer-Simpson hypothesis on which Bradley's refractions were founded. In this case (4) and (5) become

$$A=\frac{\mu_0 a}{n}\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^2r^2}=\frac{1}{n}\left(1-\frac{\mu_0 a}{b}\right)=\frac{1}{n}(1-\mu_0{}^{-n});$$

$$B=\frac{\mu_0 a}{2n}\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^2r^2}\left(1-\frac{\mu_0{}^2a^2}{\mu^2r^2}\right)$$

$$=\frac{1}{2n}\left(1-\frac{\mu_0 a}{b}\right)-\frac{1}{6n}\left(1-\frac{\mu_0{}^3a^3}{b^3}\right)$$

$$=\frac{1}{6n}\left(1-\frac{\mu_0 a}{b}\right)^2\left(2+\frac{\mu_0 a}{b}\right)=\frac{n}{6}A^2(2+\mu_0{}^{-n})$$

$$=\frac{n}{6}A^2(3-nA).$$

Hence the equation for $n$ is

$$n^2A^2-3nA+6B/A=0,$$

while the horizontal refraction is

$$C=\frac{\mu_0 a}{n}\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^2r^2}\left(1-\frac{\mu_0{}^2a^2}{\mu^2r^2}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{n}\sec^{-1}\frac{b}{\mu_0 a}=\frac{1}{n}\sec^{-1}\mu_0{}^n$$

$$=\frac{2}{n}\tan^{-1}\left(\frac{\mu_0{}^n-1}{\mu_0{}^n+1}\right)^{\frac{1}{2}}.$$

With the accepted values, $A=58''\cdot294=0\cdot0002826$ and $B/A=0\cdot001146$, it follows that $nA=0\cdot002294$, $n=8\cdot117$. Hence $\mu_0{}^n=(1-nA)^{-1}=1\cdot002299$, $\mu_0=1\cdot0002832$, and $b/a=\mu_0{}^{n+1}=1\cdot002582$. The resulting horizontal refraction is $C=28'\cdot7$, which is about 20 per cent. in defect of the true value.

5. That this result should prove unsatisfactory appears scarcely surprising, since, so far from being constant,

$$\frac{rd\mu}{d(\mu r)}=\frac{d(\log\mu)}{d(\log\mu r)}\to 0$$

as $r$ increases. It is thus more natural to assume that

$$\frac{rd\mu}{d(\mu r)}+\frac{c}{\mu r}=0, \tag{9}$$

which satisfies this obvious condition. In this case

$$A=\mu_0 ac\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^3 r^3}=\tfrac{1}{2}\mu_0 ac\left(\frac{1}{\mu_0{}^2a^2}-\frac{1}{b^2}\right); \tag{10}$$

$$\begin{aligned}B&=\tfrac{1}{2}\mu_0 ac\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^3 r^3}\left(1-\frac{\mu_0{}^2a^2}{\mu^2 r^2}\right)\\&=\tfrac{1}{2}\mu_0 ac\left\{\tfrac{1}{2}\left(\frac{1}{\mu_0{}^2a^2}-\frac{1}{b^2}\right)-\tfrac{1}{4}\mu_0{}^2a^2\left(\frac{1}{\mu_0{}^4a^4}-\frac{1}{b^4}\right)\right\}\\&=\tfrac{1}{4}A\left(1-\frac{\mu_0{}^2a^2}{b^2}\right).\end{aligned} \tag{11}$$

For the horizontal refraction (6) gives

$$\begin{aligned}C&=\mu_0 ac\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^3 r^3}\left(1-\frac{\mu_0{}^2a^2}{\mu^2 r^2}\right)^{-\frac{1}{2}}\\&=\frac{c}{\mu_0 a}\left(1-\frac{\mu_0{}^2a^2}{b^2}\right)^{\frac{1}{2}}\\&=\sqrt{(A^3/B)}=28'\ 42'',\end{aligned} \tag{12}$$

which is identical, for the same values of $A$ and $B$, with the result derived from the Bouguer-Simpson hypothesis.

6. To make another trial, let

$$\frac{rd\mu}{d(\mu r)}+\frac{c}{\mu^3 r^3}=0.$$

Then

$$A=\mu_0 ac\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^5 r^5}=\tfrac{1}{4}\mu_0 ac\left(\frac{1}{\mu_0{}^4a^4}-\frac{1}{b^4}\right);$$

$$\begin{aligned}B&=\tfrac{1}{2}\mu_0 ac\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^5 r^5}\left(1-\frac{\mu_0{}^2a^2}{\mu^2 r^2}\right)\\&=\tfrac{1}{2}\mu_0 ac\left\{\tfrac{1}{4}\left(\frac{1}{\mu_0{}^4a^4}-\frac{1}{b^4}\right)-\tfrac{1}{6}\mu_0{}^2a^2\left(\frac{1}{\mu_0{}^6a^6}-\frac{1}{b^6}\right)\right\}\\&=\tfrac{1}{24}\mu_0 ac\left(\frac{1}{\mu_0{}^2a^2}-\frac{1}{b^2}\right)\left(\frac{1}{\mu_0{}^2a^2}+\frac{1}{b^2}-\frac{2\mu_0{}^2a^2}{b^4}\right)\\&=\tfrac{1}{24}\mu_0{}^3a^3c\left(\frac{1}{\mu_0{}^2a^2}-\frac{1}{b^2}\right)^2\left(\frac{1}{\mu_0{}^2a^2}+\frac{2}{b^2}\right);\end{aligned}$$

$$\begin{aligned}C&=\mu_0 ac\int_{\mu_0 a}^{b}\frac{d(\mu r)}{\mu^5 r^5}\left(1-\frac{\mu_0{}^2a^2}{\mu^2 r^2}\right)^{-\frac{1}{2}}\\&=\tfrac{1}{3}c\left(\frac{2}{\mu_0{}^2a^2}+\frac{1}{b^2}\right)\left(\frac{1}{\mu_0{}^2a^2}-\frac{1}{b^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Hence
$$\frac{A^3}{B}=\tfrac{3}{8}c^2\left(\frac{1}{\mu_0{}^2a^2}-\frac{1}{b^2}\right)\left(\frac{1}{\mu_0{}^2a^2}+\frac{1}{b^2}\right)^3\left(\frac{1}{\mu_0{}^2a^2}+\frac{2}{b^2}\right)^{-2}$$
$$=\tfrac{2\,7}{8}C^2\left(\frac{1}{\mu_0{}^2a^2}+\frac{1}{b^2}\right)^3\left(\frac{1}{\mu_0{}^2a^2}+\frac{2}{b^2}\right)^{-1}\left(\frac{2}{\mu_0{}^2a^2}+\frac{1}{b^2}\right)^{-2};$$

and again, because $b$ does not differ greatly from $\mu_0 a$, $C^2$ is nearly equal to $A^3/B$. It does not seem possible to raise the value of the horizontal refraction by any simple hypothesis on these lines.

7. The type of atmosphere suggested in § 5 seems the simplest possible, and the difficulty in reconciling the observed horizontal refraction with the law at small zenith distances suggests a different method. Instead of trying to invent a new model, let us imagine two co-existing atmospheres of the same type. The effects, being small, can be added together.

Thus
$$A_1+A_2=A=58''{\cdot}294,$$
$$B_1+B_2=B=\ 0''{\cdot}0668,$$
$$A_1\sqrt{\frac{A_1}{B_1}}+A_2\sqrt{\frac{A_2}{B_2}}=C=2189'',$$

are three equations immediately available. But a fourth is necessary.

It is naturally tempting to identify the constituents with oxygen and nitrogen. Now the refractive indices are roughly

$$O:\ 1{\cdot}000273,\qquad N:\ 1{\cdot}000300,\qquad \text{air}:\ 1{\cdot}000294,$$

and the value of $A$ is approximately $\mu_0-1$. As these indices contribute $\frac{8}{9}$ $O$ and $\frac{1}{9}$ $N$ to the index for the mixture, $A$ may be divided into the corresponding fractions

$$A_1=12''{\cdot}954,\qquad A_2=45''{\cdot}340,$$

and on this supposition the above equations can be solved. There are in fact two solutions:

$$B_1=0''{\cdot}00223,\qquad B_1=0''{\cdot}0646;$$

or
$$B_1=0''{\cdot}0426,\qquad B_2=0''{\cdot}0242.$$

8. The result of combining (7) with (9) is

$$R=\mu_0 ac\sin\zeta\int_{\mu a_0}^{b}\frac{d(\mu r)}{\mu^2r^2(\mu^2r^2-\mu_0{}^2a^2\sin^2\zeta)^{\frac{1}{2}}}$$
$$=\frac{c}{\mu_0 a\sin\zeta}\left\{\left(1-\frac{\mu_0{}^2a^2\sin^2\zeta}{b^2}\right)^{\frac{1}{2}}-\cos\zeta\right\}.$$

But, from (10) and (11),
$$2A\,.\,\mu_0 a=c\left(1-\frac{\mu_0{}^2a^2}{b^2}\right)=\frac{4Bc}{A}.$$

Hence
$$R=\frac{A^2}{2B\sin\zeta}\left[\left\{1-\left(1-\frac{4B}{A}\right)\sin^2\zeta\right\}^{\frac{1}{2}}-\cos\zeta\right]$$
$$=\frac{2A}{\left(\cot^2\zeta+\dfrac{4B}{A}\right)^{\frac{1}{2}}+\cot\zeta}.\qquad(13)$$

With the numbers found above, therefore,

$$R=\frac{25''\cdot908}{(\cot^2\zeta+0\cdot00069)^{\frac{1}{2}}+\cot\zeta}+\frac{90''\cdot680}{(\cot^2\zeta+0\cdot00570)^{\frac{1}{2}}+\cot\zeta} \qquad (I)$$

$$=37630''(\cot^2\zeta+0\cdot00069)^{\frac{1}{2}}+15910''(\cot^2\zeta+0\cdot00570)^{\frac{1}{2}}-53540''\cot\zeta, \quad (14)$$

with the possible alternative,

$$R=\frac{25''\cdot908}{(\cot^2\zeta+0\cdot01315)^{\frac{1}{2}}+\cot\zeta}+\frac{90''\cdot680}{(\cot^2\zeta+0\cdot002135)^{\frac{1}{2}}+\cot\zeta}. \qquad (II)$$

These formulæ are compared with the mean refractions (30 in., 50° F.) taken from Chauvenet, vol. ii, and given in the second column under *M*, in the following table. On this basis it seems fair to reject the solution II at once.

TABLE I

*Comparison of Formulæ with Mean Refractions M*

| $\zeta$ | *M* | *O* | *N* | I | *M* − I | II | *M* − II |
|---|---|---|---|---|---|---|---|
| ° | ′ ″ | ″ | ″ | ′ ″ | ″ | ′ ″ | ″ |
| 90 | 36 29 | 987 | +1201 = | 36 28 | + 1 | 36 28 | + 1 |
| 89 | 24 54 | 528·7 | + 954·6 | 24 43 | +11 | 25 50 | −56 |
| 88 | 18 26 | 329·6 | + 767·8 | 18 17 | + 9 | 19 5 | −39 |
| 87 | 14 25 | 233·4 | + 628·3 | 14 22 | + 3 | 14 47 | −22 |
| 86 | 11 44 | 179·1 | + 524·8 | 11 44 | 0 | 11 57 | −13 |
| 85 | 9 52 | 144·8 | + 446·7 | 9 52 | 0 | 9 58 | − 6 |
| 84 | 8 28·1 | 121·3 | + 386·7 | 8 28·0 | + 0·1 | | |
| 83 | 7 23·8 | 104·3 | + 339·8 | 7 24·1 | − 0·3 | | |
| 82 | 6 33·3 | 91·4 | + 302·3 | 6 33·7 | − 0·4 | | |
| 81 | 5 52·6 | 81·2 | + 271·7 | 5 52·9 | − 0·3 | | |
| 80 | 5 19·2 | 73·1 | + 246·4 | 5 19·5 | − 0·3 | | |
| 75 | 3 34·1 | 48·2 | + 166·0 | 3 34·2 | − 0·1 | | |

9. The standard of comparison, which happened to be the most easily accessible, is not likely to be immaculate, and a more modern one might be substituted; but it serves its purpose. Without any attempt to adjust the four constants it contains, formula (I) or (14) appears a fair representation of the refraction over its whole extent. For the effective height of an atmosphere (11) gives

$$\frac{b}{a}=\mu_0\left(1-\frac{4B}{A}\right)^{-\frac{1}{2}},$$

and for the two constituents imagined,

$$b_1=1\cdot000061(1-0\cdot00069)^{-\frac{1}{2}}a;$$
$$b_1-a=0\cdot0001410a=2\cdot6 \text{ km.};$$
$$b_2=1\cdot000233(1-0\cdot00570)^{-\frac{1}{2}}a;$$
$$b_2-a=0\cdot00309a=19\cdot7 \text{ km.}$$

That the effect of the " nitrogen " atmosphere should be mainly produced

in the lowest 20 km. seems not unreasonable, but that the effect of the " oxygen " constituent should be confined chiefly to the lowest 3 km. cannot be accepted literally. Both from theory and from observation it is known that the oxygen-nitrogen ratio is almost uniform in the lower stratum of the atmosphere.

10. It must always be difficult to argue from the observed facts of refraction to the real structure of the atmosphere. For instance, if the Earth's surface were plane and the atmospheric strata horizontal, a single homogeneous layer would represent the facts exactly. Similarly, in the case of the spherical Earth, Professor Turner, in the paper already quoted and remembered only after the above discussion had been made, showed that the facts could be represented quite well by two homogeneous shells. Here a much more realistic model of the atmosphere has been used, but no agreement with observation, even were it perfect, would prove that the model is actually a true one. In Turner's Table I (*loc. cit.*) there is a certain resemblance with the results in the above table (§ 8) when his inner shell is identified with the *O* constituent and his outer shell with the *N* constituent. This is to some extent due to a chance circumstance : in the absence of any guiding principle he found it " convenient " to assign to the two shells equal contributions to the horizontal refraction, and there is an approach to this equality in the present case. When a third shell was added the similar guess was less lucky, and the comparative failure to secure improvement by this means seems to be explained.

On one point there is certainly agreement, in spite of the totally different method and point of view. The present discussion equally emphasises (in Turner's words) " the fact that a serious part of the refraction takes place near the Earth, and not high in the atmosphere." Here the main effect is largely located below an altitude of 3 km.

As in a general way absorption goes with refraction, the advantage of choosing an altitude of the order of 2 km. for an observing station is also emphasised.

11. Having served their turn, the *O* and *N* constituents, regarded as distinct factors in the refraction, must be dismissed as pure fictions, but their relation to reality deserves a little more consideration.

In a more general form (9) can be written

$$r\frac{d\mu}{d(\mu r)}+\frac{c}{(\mu r)^p}=0.$$

When $p=0$, the integral is given by (8). When $p>0$, the integral is

$$\log\mu-\frac{c}{p(\mu r)^p}=\text{const.}$$
$$=\log\mu_0-\frac{c}{p(\mu_0 a)^p}=-\frac{c}{pb^p};$$

$b$ being, as before, the value of $r$ when $\mu=1$. The case of $p=1$ has been used in § 5, and afterwards adopted as normal on account of its convenience. The case of $p=3$ has been used in § 6 ; odd integral values of $p$ conduce to simple

integration. Now let $\mu = 1 + \nu$, $r = b(1 - x)$ ; both $\nu$ and $x$ are always small. To the first order in $\nu$,

$$\nu - \frac{c}{pr^p}(1 - p\nu) = -\frac{c}{pb^p},$$

$$\nu = \frac{c}{pb^p} \cdot \frac{b^p - r^p}{c + r^p} = \frac{cx}{c + b^p},$$

to the first order in $x$. Hence for all values of $p$, and all laws of this type, $\nu$ is sensibly proportional to $x$ and $\mu$ is a linear function of the height. This property, well known in the case of the Bouguer-Simpson law ($p = 0$), will account for the uniformity in the results of §§ 4–6.

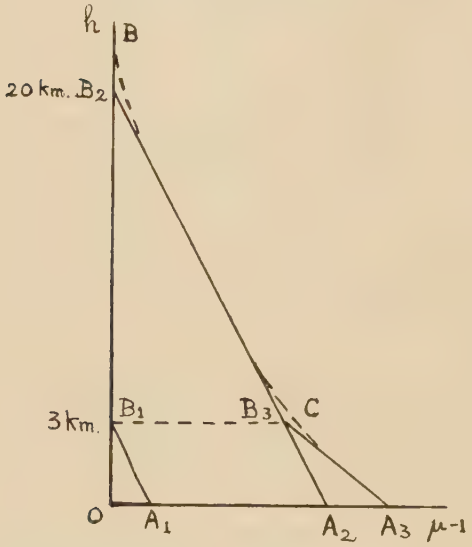



FIG. 1

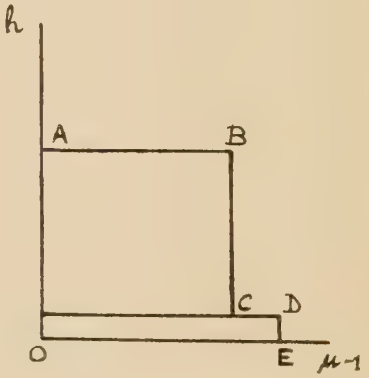



FIG. 2

12. A graphical representation will help to explain the nature of the subsequent process.

In fig. 1 the abscissæ represent $\mu - 1$ and the ordinates the height $h$. A division is first made between the part $A_1B_1$ due to the $O$ atmosphere and the part $A_2B_2$ due to the $N$ atmosphere. But by moving the line $OA_1$ into the position $A_2A_3$ and the point $B_1$ to $B_3$, the whole relation between $\mu - 1$ and $h$ can be indicated by the contour $A_3B_3B_2$ made up of two straight lines. It then seems evident that, with little violence to this rectilinear contour, a continuous curve such as $A_3CB$ might be substituted with little influence on the results but probably a nearer approach to reality. The separation into two constituent media, though it is possible and may be convenient, cannot be regarded as necessary.

In fig. 2 the corresponding contour for Turner's two homogeneous shells is roughly indicated. It is not strange that he expressed surprise at the closeness of the resulting representation. This is purely and frankly empirical ; it succeeds in expressing facts but scarcely in explaining them.

13. From the practical point of view the physical distinction between the two constituents is now seen to be simply irrelevant. It is merely a device which allows of two different gradients in the refractive index of the air at different heights. What amendment of this simple structure will be needed to account for the finer details of the refraction can be left as an open question. In the meantime it is necessary to determine the gradients more accurately by adjusting the relative proportion of the two constituents, which may still be called $O$ and $N$, though they are quite certainly one and the same gas—air. In § 7 this proportion was taken to be 2 : 7. A little experimental calculation shows that the ratio 3 : 8 is more appropriate, so that

$$A_1 = 15''{\cdot}898, \qquad A_2 = 42''{\cdot}395.$$

Corresponding to this division,

$$B_1 = 0''{\cdot}00336, \qquad B_2 = 0''{\cdot}06344,$$

and hence, after inserting these numbers in (13),

$$R = \frac{31''{\cdot}796}{(\cot^2\zeta + 0{\cdot}000846)^{\frac{1}{2}} + \cot\zeta} + \frac{84''{\cdot}790}{(\cot^2\zeta + 0{\cdot}005985)^{\frac{1}{2}} + \cot\zeta}. \qquad \text{(III)}$$

The result of comparing this formula with the same table as before is shown in similar form in Table II.

14. Refractions in the most difficult range, 85° to 90°, are now perfectly represented. In the range 75° to 85° there is a systematic trend which is no less than that shown by formula II. But clearly this is an approach to the stage when it will be necessary to make a re-determination of the constants by direct comparison with observation. It may be noticed that, without making an apparently lucky assumption like Turner's, the horizontal refraction is now divided into two essentially equal parts. Unfortunately no reason has appeared for making this assumption.

TABLE II

*Comparison of Formula III with Mean Refractions M*

| $\zeta$ | $M$ | $O$ | $N$ | III | $M$ − III |
|---|---|---|---|---|---|
| ° | ′ ″ | ″ | ″ | ′ ″ | ″ |
| 90 | 36 29 | 1093 | +1096 | =36 29 | 0 |
| 89 | 24 54 | 618·9 | + 876·3 | 24 55 | −1 |
| 88 | 18 26 | 395·6 | + 707·8 | 18 23 | +3 |
| 87 | 14 25 | 283·0 | + 581·4 | 14 24 | +1 |
| 86 | 11 44 | 218·3 | + 486·8 | 11 45 | −1 |
| 85 | 9 52 | 177·0 | + 415·1 | 9 52 | 0 |
| 84 | 8 28·1 | 148·5 | + 359·9 | 8 28·4 | −0·3 |
| 83 | 7 23·8 | 127·7 | + 316·5 | 7 24·2 | −0·4 |
| 82 | 6 33·3 | 111·9 | + 281·7 | 6 33·6 | −0·3 |
| 81 | 5 52·6 | 99·5 | + 253·4 | 5 52·9 | −0·3 |
| 80 | 5 19·2 | 89·5 | + 229·9 | 5 19·4 | −0·2 |
| 75 | 3 34·1 | 59·2 | + 155·0 | 3 34·2 | −0·1 |

Otherwise the results of Table II and the corresponding formula III for the mean refractions could have been deduced directly from the three constants $A$, $B$ and $C$ without the slightest numerical adjustment. Thus

$$A_1 + A_2 = A, \qquad B_1 + B_2 = B,$$

$$\frac{A_1^3}{B_1} = \frac{A_2^3}{B_2} = \tfrac{1}{4}C^2,$$

lead directly to

$$A_2,\ A_1 = \tfrac{1}{2}\left[A \pm \left(\frac{BC^2 - A^3}{3A}\right)^{\frac{1}{2}}\right],$$

and the rest of the solution follows. The circumstance is remarkable enough, but it appears to be wholly fortuitous. So much depends on the constant $B$ that it seems far from certain that a quite satisfactory value has been adopted.

15. To re-calculate the effective heights of the two parts of the atmosphere,

$$b_1 = 1{\cdot}000080(1 - 0{\cdot}000845)^{-\frac{1}{2}}a;$$
$$b_1 - a = 0{\cdot}000503a = 3{\cdot}2 \text{ km.};$$
$$b_2 = 1{\cdot}000214(1 - 0{\cdot}005987)^{-\frac{1}{2}}a;$$
$$b_2 - a = 0{\cdot}003221a = 20{\cdot}5 \text{ km.}$$

The gradients of the refractive indices in the two components are therefore $0{\cdot}000080/3{\cdot}2 = 0{\cdot}000025$ per km. and $0{\cdot}000214/20{\cdot}5 = 0{\cdot}0000105$ per km., or

0·0000355 per km. below 3·2 km.
0·0000105 ,, between 3·2 and 20·5 km.

are the mean gradients. The corresponding (isothermal) barometric gradient in the lower stratum is

$$\frac{0{\cdot}0000355}{0{\cdot}000294} \times 760 = 92 \text{ mm. per km.},$$

or 1·1 in. per 1000 ft., which is of the right order of magnitude in the lowest two miles. The height of the corresponding homogeneous atmosphere is $\frac{1}{2}(\frac{3}{11} \times 3{\cdot}2 + \frac{8}{11} \times 20{\cdot}5) = 7{\cdot}89$ km., while 7·99 km. is the true figure for 760 mm., 0° C. Nearly the whole atmosphere is thus taken into account.

16. From the vertical gradients in the refractive index the atmospheric density at different heights can be approximately deduced. The results are given in Table III and compared with figures given by Mr. W. H. Dines.* It is unfortunate that the observed data end at 14 km., because, as a graphical representation suggests clearly enough, agreement is to be expected again at 16 or 17 km. On the whole the agreement is rough. For all that it is good enough to explain the nature of the process by which relative displacements of the natural strata are admitted with mutually compensating effect on the refractions.

* Quoted in *Dictionary of Applied Physics*, **3**, 60.

TABLE III

| Height o km. | Density Calc. (1251) gm./m.$^3$ | Density Obs. (High) | Density Obs. (Low) | Mean—Calc. |
|---|---|---|---|---|
| 1 | 1100 | 1135 | 1105 | + 20 |
| 2 | 949 | 1010 | 997 | + 55 |
| 3 | 799 | 906 | 898 | +103 |
| 4 | 733 | 818 | 808 | + 80 |
| 5 | 688 | 734 | 724 | + 41 |
| 6 | 644 | 658 | 648 | + 9 |
| 7 | 599 | 589 | 582 | − 13 |
| 8 | 555 | 530 | 516 | − 32 |
| 9 | 511 | 472 | 446 | − 52 |
| 10 | 466 | 427 | 389 | − 58 |
| 11 | 421 | 350 | 333 | − 79 |
| 12 | 377 | 316 | 285 | − 77 |
| 13 | 333 | 272 | 245 | − 74 |
| 14 | 288 | 232 | 210 | − 67 |

17. Comparisons with the tabular refractions have been given only for zenith distances greater than 70°. The reason will be clear. The formula (13) can be written

$$R=\frac{A^2}{2B}\cot\zeta\left\{\left(1+\frac{4B}{A}\tan^2\zeta\right)^{\frac{1}{2}}-1\right\}$$

$$=A\tan\zeta-B\tan^3\zeta+\frac{2B^2}{A}\tan^5\zeta-\frac{5B^3}{A^2}\tan^7\zeta+\cdots$$

so long as $\zeta$ is sufficiently small. When the atmosphere is divided into two parts, this becomes

$$R=A\tan\zeta-B\tan^3\zeta+2\left(\frac{B_1^2}{A_1}+\frac{B_2^2}{A_2}\right)\tan^5\zeta$$
$$-5\left(\frac{B_1^3}{A_1^2}+\frac{B_2^3}{A_2^2}\right)\tan^7\zeta+\cdots$$

the expansion being legitimate provided that

$$\tan^2\zeta<\frac{A_1}{4B_1}<\frac{A_2}{4B_2}.$$

With $\zeta<70°$ the expansion can be restricted to the first two terms, and as these are based on accepted constants no special comparison is required.

A convenient form can also be given for low altitudes, namely,

$$R=\left(\frac{A_1^3}{B_1}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\phi_1+\left(\frac{A_2^3}{B_2}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\phi_2,$$

where

$$\tan \phi_1 = \left(\frac{4B_1}{A_1}\right)^{\frac{1}{2}} \tan \zeta, \qquad \tan \phi_2 = \left(\frac{4B_2}{A_2}\right)^{\frac{1}{2}} \tan \zeta,$$

so that $\frac{1}{2}\phi_1 = \frac{1}{2}\phi_2 = 45^\circ$ when $\zeta = 90^\circ$. When $\phi_1$, $\phi_2$ are small, which does not require $\zeta$ to be particularly small, the form $R = A \tan \zeta$ is reproduced.

Numerically the formulæ

$$\tan \phi_1 = [8{\cdot}46375] \tan \zeta, \qquad \tan \phi_2 = [8{\cdot}88854] \tan \zeta,$$
$$R = [3{\cdot}03862] \tan \tfrac{1}{2}\phi_1 + [3{\cdot}03981] \tan \tfrac{1}{2}\phi_2,$$

will represent the Bessel refractions much better than the latter agree with the Pulkowa refractions. Naturally the numerical constants must be adapted to the particular system to be represented.

## *Summary*

18. (i) The general run of the refraction in minutes is expressed by $R = \tan(\zeta^\circ - \zeta')$, or $\tan(1 - \frac{1}{60})\zeta$, with about the same accuracy as can be derived from a single homogeneous Cassini shell (§ 1).

(ii) For moderate zenith distances the form $R = A \tan \zeta - B \tan^3 \zeta$ is immediately suggested by the general theory without any hypothesis regarding the structure of the atmosphere (§§ 2, 3).

(iii) For a class of assumed functional relations between $\mu$ and $r$, including the Bouguer-Simpson hypothesis, the horizontal refraction $C$ is connected with $A$ and $B$ by the equation $C^2 = A^3/B$ either exactly or approximately (§§ 4–6).

(iv) The observed value of $C$ can be connected with the known values of $A$ and $B$ by attributing the refraction to two constituents in the atmosphere, but these cannot be identified with oxygen and nitrogen because the main effect is produced in the lower strata where the ratio of these gases is almost constant (§§ 7–9).

(v) Owing to the small range of $r$ in terms of the Earth's radius, the assumption that $\mu$ is a regular function of $r$ virtually implies that $\mu$ is a linear function of $r$. This allows for only one vertical gradient in the value of $\mu$, which is inadequate to explain the facts of refraction (§§ 10, 11).

(vi) The division of the atmosphere into two parts is a convenient way of allowing for two gradients. This interpretation is fairly confirmed by a comparison of the gradients derived from the refractions with the densities of the atmosphere observed at heights up to 14 km. (§§ 12–16).

(vii) The result is to connect the form

$$R = A \tan \zeta - B \tan^3 \zeta,$$

which is convenient at moderate zenith distances, with the more general form

$$R = a_1 \tan \tfrac{1}{2}\{\tan^{-1}(b_1 \tan \zeta)\} + a_2 \tan \tfrac{1}{2}\{\tan^{-1}(b_2 \tan \zeta)\},$$

which extends to the horizon. Values of the constants have been found giving good agreement with Bessel's mean refractions (§§ 14, 17).

*Photographic Observations of Double Stars.* By W. M. Smart, M.A., D.Sc.

(*Communicated by the Director, Cambridge Observatory.*)

1. The results given in this paper have been obtained by measurement of photographs taken with the Sheepshanks Equatorial (Coudé), focal length 19·3 feet. The area covered by a plate is $1\frac{1}{2}^\circ \times 1\frac{1}{2}^\circ$ and the scale is 1 mm. $= 35''{\cdot}2$. With a small number of exceptions, the plates which have provided the measures of the double stars were taken for quite different purposes ; they are mostly parallax and proper-motion plates, the earliest epoch being 1900. The plates were oriented in the measuring machine by means of the réseau lines or, in a smaller number of instances, by means of star-trails. The measures, in rectangular co-ordinates, were made by Mr. H. E. Green, and in a few instances by Mr. Alan Fletcher.

Unless the double star concerned happened also to be a parallax star, the exposures were not necessarily the most suitable that could be given, to achieve the best results, in a series of observations devoted specifically to the measurement of double stars. Also, the double stars measured are in all parts of the plates, a circumstance which reduces somewhat the most favourable conditions for securing suitable images. Despite these disadvantages it would seem that the results are comparable in accuracy with the best visual observations even when the separation is as small as 3 or 4 seconds of arc.

2. If $\Delta x$ and $\Delta y$ denote the measures, parallel to the $x$ and $y$ axes, relating to the components of a double star at the centre of the plate, the distance $\rho$ and the position angle $\theta$ are given by

$$\rho^2 = (\Delta x)^2 + (\Delta y)^2, \tag{1}$$

$$\tan\theta = \Delta x/\Delta y. \tag{2}$$

In the first of these formulæ $\Delta x$ and $\Delta y$ are supposed to be expressed in seconds of arc. If the double star is not at the centre of the plate, $\rho$ is still given with sufficient accuracy by (1), but the position angle is now given by

$$\theta = \tan^{-1}(\Delta x/\Delta y) + \Delta\theta, \tag{3}$$

where $\Delta\theta$ in degrees is defined (for the Sheepshanks telescope) by

$$\Delta\theta = 0^\circ{\cdot}049\xi_1 \tan D. \tag{4}$$

In (4), $D$ is the declination of the centre of the plate and $\xi_1$ is the star's standard co-ordinate, parallel to the equator, expressed in units of the réseau scale. The formula (4) can be derived simply as follows.* Let $(\xi, \eta)$ be the standard co-ordinates of the brighter component $S$ of a double star (right ascension $\alpha$, declination $\delta$) with reference to a plate centred at

* For a more elaborate discussion, see H. C. Plummer, *M.N.*, **69**, 100, 1908.

$O$ (right ascension $A$, declination $D$). The projection on the plate of the meridian through $O$ defines the $\eta$-axis $OY$, and the $\xi$-axis $OX$ is drawn through $O$ perpendicular to $OY$. These are the axes of reference. The position angle of the double star concerned is defined with reference to the meridian through $S$, or alternatively, with reference to the parallel of declination $\delta$. On the plate the projection of the meridian through $S$ is a straight line $SY'$ not quite parallel to the axis $OY$, and the projection of the parallel of declination $\delta$ is a parabola, the tangent $SX'$ at $S$ to the curve making a small angle with the axis $OX$. We shall show that $SX'$ and $SY'$ are perpendicular and are derived by rotating the axes $OX$ and $OY$ through a positive angle given in absolute measure by $\xi \tan D$.

(*a*) Let $p$ be the projection of the pole on the plate. The standard co-ordinates of $p$ are (0, cot $D$). The axis $SY'$ is the straight line joining $S$ to $p$, and if $\overrightarrow{SY'}$ makes an angle $90° + \beta$ with $\overrightarrow{OX}$,

$$\tan \beta = \frac{\xi}{\cot D - \eta}$$

or, to the first order in the small quantities $\xi$ and $\eta$,

$$\beta = \xi \tan D. \tag{5}$$

(*b*) We have the formulæ *

$$\cot \delta \cos (\alpha - A) = \frac{1 - \eta \tan D}{\eta + \tan D}, \tag{6}$$

$$\cot \delta \sin (\alpha - A) = \frac{\xi \sec D}{\eta + \tan D}. \tag{7}$$

Squaring (6) and (7) and adding, we eliminate $(\alpha - A)$ and so obtain the equation to the curve representing the projection of the parallel of declination $\delta$ ; it is

$$\cot^2 \delta (\eta + \tan D)^2 = \xi^2 \sec^2 D + (1 - \eta \tan D)^2. \tag{8}$$

Let the curve cut the axis $OY$ at (0, $\eta_0$). Then for this point $(\alpha - A) = 0$, and therefore from (6)

$$\eta_0 = \tan (\delta - D). \tag{9}$$

Write $\eta = \eta_0 + \Delta\eta$ in (8) and, neglecting $\Delta\eta^2$, we obtain

$$2\Delta\eta[(\eta_0 + \tan D) \cot^2 \delta + \tan D(1 - \eta_0 \tan D)] = \xi^2 \sec^2 D,$$

from which, using (9),

$$\Delta\eta = \tfrac{1}{2}\xi^2 \tan \delta.$$

Hence the equation of the parallel of declination is

$$\eta = \eta_0 + \tfrac{1}{2}\xi^2 \tan \delta. \tag{10}$$

If $\gamma$ is the angle which the tangent at $S$ makes with the axis $OX$,

$$\tan \gamma \equiv \frac{d\eta}{d\xi} = \xi \tan \delta$$

* W. M. Smart, *Spherical Astronomy*, p. 284, 1931.

or, with sufficient accuracy,

$$\gamma = \xi \tan D. \qquad (11)$$

Thus it is seen from (5) and (11) that $SX'$ and $SY'$ are perpendicular. Hence to obtain the position angle $\theta$ we require to add to $\tan^{-1}(\Delta x/\Delta y)$ the quantity $\Delta\theta \equiv \xi \tan D$. Expressing $\Delta\theta$ in degrees and $\xi$ in réseau units (1 réseau unit = 5 mm. = 176″) we obtain formula (4) for plates taken with the Sheepshanks telescope.

This correction for position angle has been applied to all the measures.

3. In the Cambridge list of stars there are 56 with visual measures at nearly contemporary epochs in Burnham's General Catalogue ($\beta$.G.C.), and 13 in a list recently published * by the Observatory of Lyon, the epochs of which, however, are generally 10 to 20 years later than the Cambridge epochs. Stars with recognised orbital motion or of the 61 Cygni type have been avoided in making the comparison shown in Table I. In this table the stars are arranged in groups according to the separation $\rho$. $\Delta\theta$ and $\Delta\rho$ denote the average discrepancy in position angle and distance taken without regard to sign. $\Delta\rho'$ denotes the algebraic mean of the differences $\rho$(Cambridge) $-\rho(\beta$.G.C. or Lyon). The quantities $\Delta\theta$, $\Delta\rho$ and $\Delta\rho'$ contain, in addition to the errors of the visual and photographic observations, the unknown effects of relative motion—these effects may be due to previously unsuspected orbital motion or the relative proper motion of the components —in the interval between the epochs of observation. For a comparison

Table I

| | $\beta$.G.C. | | | | Lyon | | | |
|---|---|---|---|---|---|---|---|---|
| | No. of Stars | $\Delta\theta$ | $\Delta\rho$ | $\Delta\rho'$ | No. of Stars | $\Delta\theta$ | $\Delta\rho$ | $\Delta\rho'$ |
| | | ° | ″ | ″ | | ° | ″ | ″ |
| $3'' < \rho < 5''$ | 13 | 1·2 | 0·16 | −0·07 | | | | |
| $5'' < \rho < 10''$ | 18 | 0·8 | 0·14 | +0·04 | | | | |
| $10'' < \rho < 20''$ | 8 | 0·8 | 0·16 | +0·08 | 5 | 0·9 | 0·23 | −0·05 |
| $\rho > 20''$ | 17 | 0·6 | 0·21 | +0·13 | 8 | 0·6 | 0·16 | −0·04 |

test to be thoroughly searching, the photographic and visual epochs should be the same, but despite the additional uncertainty introduced by non-compliance with this condition, the comparisons shown in Table I appear to be as satisfactory as similar comparisons between two sets of visual observations. The quantities $\Delta\rho'$ under "$\beta$.G.C." apparently indicate a very small systematic correction of the type encountered in one or two previous comparisons of photographic and visual observations. There is no indication of such a correction in the column under "Lyon," but in this instance there is not much material on which to base a judgment. As one would expect, the quantities $\Delta\theta$ decrease as $\rho$ increases.

* *Bulletin de l'Obs. de Lyon*, **13**, No. 9, 1931 September.

4. In Table II the β.G.C. stars of Table I are grouped according to the difference $\Delta m$ between the magnitudes of the components of the double stars. $\Delta\theta$, $\Delta\rho$ and $\Delta\rho'$ have the same significance as in section 3. The last column does not reveal any systematic tendencies.

TABLE II

| | No. of Stars | $\Delta\theta$ | $\Delta\rho$ | $\Delta\rho'$ |
|---|---|---|---|---|
| | | ° | ″ | ″ |
| $0 < \Delta m < 1^m{\cdot}0$ | 24 | 0·8 | 0·13 | +0·06 |
| $1^m{\cdot}0 < \Delta m < 2^m{\cdot}0$ | 14 | 1·1 | 0·18 | −0·03 |
| $\Delta m > 2^m{\cdot}0$ | 18 | 0·8 | 0·19 | +0·07 |

5. The most exhaustive investigation of photographic measures of double stars was made in the years 1914–1919 by E. Hertzsprung.* There are five stars common to the present list and Hertzsprung's list, all in relative motion. Curves showing the variation of $\theta$ and $\rho$ with the time were plotted from the visual observations in β.G.C. and Hertzsprung's photographic measures. The values of $\theta$ and $\rho$ corresponding to the Cambridge epochs were read from the curves and compared with the Cambridge measures. Table III shows the comparison.

TABLE III

| β.G.C. | Cambridge | | | From Curves | | $\Delta\theta$ | $\Delta\rho$ |
|---|---|---|---|---|---|---|---|
| | Epoch 1900 + | $\theta$ | $\rho$ | $\theta$ | $\rho$ | | |
| | | ° | ″ | ° | ″ | ° | ″ |
| 4122 | 4·18 | 223·3 | 5·53 | 223·8 | 5·54 | −0·5 | −0·01 |
| 4972 | 7·08 | 67·0 | 19·12 | 66·5 | 19·13 | +0·5 | −0·01 |
| 6243 | 4·07 | 326·8 | 5·79 | 328·0 | 5·95 | −1·2 | −0·16 |
| 9137 | 4·57 | 218·0 | 9·31 | 218·0 | 9·34 | 0·0 | −0·03 |
| | 26·52 | 215·2 | 8·84 | 215·6 | 8·84 | −0·4 | 0·00 |
| 11483 | 7·86 | 280·2 | 7·01 | 281·8 | 6·93 | −1·6 | +0·08 |

The average numerical discordance in position angle is 0°·7 and in distance 0″·05. These quantities are of the same order of magnitude as in Tables I and II.

6. I have not undertaken the labour of computing the probable errors from the separate measures in all cases, but the following results for four representative stars chosen at random indicate in general the kind of accuracy obtained from the observations. In Table IV, $r(\theta)$ and $r(\rho)$ are the probable errors in $\theta$ and $\rho$ of a single observation, and $r_1(\theta)$ and $r_1(\rho)$ the probable errors of the arithmetical mean.

* *Publ. des Astrophys. Obs. zu Potsdam,* **24**, No. 75, 1920.

TABLE IV

| $\beta$.G.C. | $\rho$ | No. of Observations | $r(\theta)$ | $r(\rho)$ | $r_1(\theta)$ | $r_1(\rho)$ |
|---|---|---|---|---|---|---|
| | ″ | | ° | ″ | ° | ″ |
| 11830 | 3·71 | 12 | 1·6 | 0·11 | 0·4 | 0·03 |
| 11836 | 7·38 | 7 | 0·3 | 0·07 | 0·1 | 0·03 |
| 1177 | 11·16 | 4 | 0·7 | 0·14 | 0·4 | 0·08 |
| 11764 | 25·18 | 8 | 0·2 | 0·08 | 0·1 | 0·03 |

7. In the list which follows, the second column gives the designation of the star when known, the numbers referring either to $\beta$.G.C. or to B.D. The right ascensions and declinations are given for 1930·0. The position angles given in column 6 are referred to the mean equator of 1930·0, the correction in position angle due to precession being found from the formula

$$0^{\circ}\cdot014(1930 - t)\sin\alpha\sec\delta\sin\epsilon,$$

where $t$, expressed in years, is the epoch of observation, and $\epsilon$ is the obliquity of the ecliptic. The magnitudes in the penultimate column are taken from $\beta$.G.C. for stars in that catalogue and from the B.D. catalogue for corresponding stars. As regards the anonymous stars, the magnitudes are estimates from the photographic plates.

*List of Double Stars*

| No. | Star ($\beta$.G.C. or B.D.) | R.A. (1930) | Dec. (1930) | Epoch 1900 + | Position Angle | Distance | Magnitudes | No. of Observations |
|---|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | | ° | ″ | | |
| 1 | 26° 4739 | 0 0·6 | 26 37 | 7·98 | 160·2 | 13·43 | 9·0 : 10·5 | 8 |
| | | | | 26·75 | 161·0 | 13·41 | ... | 4 |
| 2 | 12736 | 0 1·4 | 45 18 | 7·84 | 342·1 | 3·96 | 8·0 : 9·0 | 10 |
| 3 | 44° 4551 | 0 1·6 | 44 50 | 7·84 | 234·9 | 21·59 | 6·3 : 10·5 | 8 |
| 4 | 12740 | 0 1·9 | 45 26 | 7·84 | 133·2 | 4·78 | 8·3 : 8·4 | 12 |
| 5 | 14° 201 | 1 15·5 | 15 16 | 10·81 | 65·3 | 4·99 | 9·4 : 11·0 | 1 |
| 6 | 802 | 1 29·8 | 45 38 | 0·95 | 318·1 | 10·03 | 8·6 : 9·0 | 15 |
| | | | | 25·88 | 319·1 | 9·81 | ... | 5 |
| 7 | 818 | 1 31·9 | 46 0 | 0·95 | 43·7 | 11·55 | 9·8 : 11·0 | 12 |
| | | | | 25·88 | 44·6 | 11·87 | ... | 5 |
| 8 | 46° 405 | 1 34·3 | 46 46 | 0·95 | 70·2 | 15·56 | 9·2 : 11·0 | 13 |
| | | | | 25·88 | 70·3 | 15·47 | ... | 5 |
| 9 | 52° 397 | 1 34·9 | 52 37 | 0·89 | 42·1 | 9·86 | 9·4 : 11·0 | 19 |
| | | | | 23·83 | 41·9 | 9·76 | ... | 8 |
| 10 | 45° 405 | 1 35·4 | 45 41 | 0·95 | 145·2 | 5·88 | 9·2 : 11·5 | 21 |
| | | | | 24·55 | 147·0 | 5·88 | ... | 17 |
| 11 | 905 | 1 40·8 | 43 21 | 0·97 | 125·0 | 5·33 | 8·0 : 8·2 | 12 |
| | | | | 25·88 | 124·9 | 5·17 | ... | 6 |
| 12 | ... | 1 47·6 | 54 9 | 8·85 | 170·2 | 13·36 | 9·5 : 12·0 | 2 |

*List of Double Stars*—continued

| No. | Star ($\beta$.G.C. or B.D.) | R.A. (1930) | Dec. (1930) | Epoch 1900 + | Position Angle | Distance | | Magnitudes | No. of Observations |
|---|---|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | | ° | ″ | | | |
| 13 | 1025 | 1 53·8 | 41 3 | 0·99 | 75·0 | 23·98 | | 8·0 : 10·0 | 3 |
| | | | | 25·88 | 77·5 | 25·05 | | ... | 6 |
| 14 | 54° 428 | 1 54·1 | 54 57 | 0·87 | 284·2 | 26·53 | | 9·6 : 9·8 | 14 |
| | | | | 25·75 | 283·8 | 26·35 | | ... | 6 |
| 15 | 54° 433 | 1 55·8 | 55 14 | 8·85 | 144·5 | 16·52 | | 9·7 : 10·7 | 8 |
| 16 | 54° 456 | 2 1·0 | 55 13 | 8·85 | 16·4 | 11·37 | | 9·7 : 11·0 | 3 |
| 17 | ... | 2 1·1 | 54 37 | 8·85 | 85·2 | 16·47 | | 10·8 : 11·5 | 2 |
| 18 | 1095 | 2 2·6 | 54 47 | 0·85 | 216·1 | 8·88 | | 10·0 : 11·0 | 10 |
| | | | | 8·85 | 215·9 | 8·97 | | ... | 7 |
| | | | | 25·86 | 217·1 | 8·63 | | ... | 4 |
| 19 | 54° 462 | 2 3·1 | 55 5 | 8·85 | 325·7 | 11·22 | | 9·2 : 11·0 | 4 |
| 20 | 1177 | 2 12·9 | 37 10 | 1·00 | 356·8 | 11·16 | | 8·3 : 10·7 | 4 |
| 21 | 1214 | 2 17·0 | 37 6 | 1·01 | 136·8 | 2·85 | | 8·5 : 9·0 | 2 |
| 22 | 1255 | 2 21·6 | 53 58 | 0·83 | 346·8 | 6·73 | | 8·2 : 8·7 | 4 |
| | | | | 25·86 | 347·6 | 6·68 | | ... | 3 |
| 23 | 1274 | 2 24·3 | 53 33 | 0·82 | 36·7 | 4·74 | | 9·5 : 10·5 | 32 |
| | | | | 25·88 | 37·9 | 4·48 | | ... | 8 |
| 24 | 1281 | 2 26·0 | 52 52 | 0·82 | 121·7 | 3·31 | | 8·2 : 8·2 | 19 |
| | | | | 25·90 | 121·7 | 3·18 | | ... | 5 |
| 25 | 1290 | 2 27·3 | 53 57 | 0·82 | 354·5 | 8·51 | | 9·5 : 10·0 | 24 |
| 26 | 1393 | 2 39·3 | 48 56 | 0·78 | 298·8 | 17·44 | | 4·2 : 10·0 | 1 |
| 27 | 48° 756 | 2 42·2 | 48 49 | 0·78 | 341·3 | 7·38 | | 8·5 : 10·5 | 7 |
| | | | | 25·88 | 341·8 | 7·40 | | ... | 4 |
| 28 | 1426 | 2 44·0 | 48 54 | 0·78 | 288·6 | 25·15 | | 8·0 : 11·0 | 17 |
| | | | | 25·88 | 289·0 | 25·42 | | ... | 4 |
| 29 | 43° 757 | 3 29·1 | 43 37 | 1·70 | 286·4 | 31·80 | AB | 9·0 : 10·0 | 4 |
| | | | | 25·06 | 286·2 | 31·60 | AB | ... | 4 |
| | | | | 1·70 | 341·4 | 14·00 | AC | 9·0 : 11·0 | 4 |
| | | | | 25·06 | 341·4 | 14·26 | AC | ... | 4 |
| 30 | 43° 762 | 3 30·1 | 43 54 | 1·70 | 67·2 | 21·47 | | 10·0 : 11·0 | 4 |
| | | | | 25·06 | 65·6 | 21·30 | | ... | 2 |
| 31 | 2379 | 4 46·0 | 6 50 | 7·76 | 3·5 | 95·81 | | 7·0 : 8·5 | 8 |
| 32 | 2441 | 4 54·9 | 30 54 | 15·85 | 56·2 | 4·27 | | 8·0 : 10·5 | 11 |
| 33 | 2449 | 4 56·9 | 62 55 | 7·78 | 119·5 | 12·63 | | 8·5 : 8·7 | 8 |
| 34 | 2451 | 4 56·9 | 62 59 | 7·78 | 210·6 | 33·00 | | 7·0 : 7·3 | 8 |
| 35 | ... | 4 59·2 | 41 32 | 7·50 | 342·0 | 7·68 | | 10·0 : 11·0 | 8 |
| 36 | ... | 5 3·1 | 40 41 | 7·50 | 130·5 | 9·21 | | 11·0 : 11·2 | 2 |
| 37 | ... | 5 12·1 | 39 42 | 7·16 | 265·1 | 12·32 | | 10·0 : 12·0 | 5 |
| 38 | ... | 5 13·5 | 40 16 | 7·16 | 81·3 | 9·93 | | 11·0 : 11·2 | 5 |

*List of Double Stars*—continued

| No. | Star ($\beta$.G.C. or B.D.) | R.A. (1930) | Dec. (1930) | Epoch 1900 + | Position Angle | Distance | Magnitudes | No. of Observations |
|---|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | | ° | ″ | | |
| 39 | ... | 5 13·6 | 40 16 | 11·36 | 82·7 | 9·77 | 10·0 : 11·0 | 7 |
| 40 | ... | 5 14·9 | 40 0 | 7·16 | 98·7 | 14·24 | 9·5 : 12·0 | 6 |
| 41 | ... | 5 14·9 | 40 50 | 11·36 | 128·2 | 10·07 | 9·8 : 11·0 | 4 |
| 42 | 39° 1260 | 5 15·7 | 39 34 | 7·16 | 267·0 | 16·05 | 9·2 : 11·0 | 7 |
| 43 | ... | 5 16·0 | 41 16 | 11·36 | 240·5 | 13·57 | 10·0 : 10·2 | 7 |
| 44 | 2653 | 5 16·6 | 40 46 | 7·16 | 89·3 | 6·28 | 9·3 : 9·4 | 8 |
| | | | | 11·36 | 91·6 | 6·36 | | 8 |
| 45 | 2663 | 5 17·2 | 40 43 | 11·36 | 178·3 | 2·50 | 9·0 : 9·2 | 12 |
| 46 | 2685 | 5 19·3 | 41 4 | 11·36 | 305·7 | 12·41 | 9·5 : 12·0 | 7 |
| 47 | 2784 | 5 27·4 | −4 18 | 7·14 | 337·8 | 27·16 | 8·0 : 11·0 | 8 |
| 48 | 2786 | 5 29·0 | 51 53 | 7·16 | 106·0 | 4·74 | 8·0 : 10·5 | 7 |
| 49 | ... | 5 30·9 | 51 17 | 7·16 | 358·8 | 5·33 | 10·0 : 11·5 | 4 |
| 50 | 2836 | 5 31·9 | −4 44 | 11·07 | 219·0 | 14·12 | 9·0 : 9·5 | 1 |
| 51 | 2837 | 5 31·9 | −5 26 | 11·07 | 129·2 | 13·08 | AC 7·0 | 1 |
| | | | | | 161·1 | 16·38 | BC 8·0 | |
| | | | | | 241·2 | 13·40 | DC 4·7 | |
| | | | | | | | 6·3 | |
| 52 | ... | 5 34·3 | 51 43 | 7·16 | 223·0 | 7·18 | 9·4 : 9·8 | 8 |
| 53 | ... | 5 35·7 | 51 9 | 7·16 | 223·6 | 19·58 | 10·5 : 11·5 | 4 |
| 54 | 11° 1223 | 6 30·1 | 11 28 | 11·08 | 160·0 | 10·34 | 8·7 : 11·5 | 2 |
| 55 | 11° 1232 | 6 31·1 | 11 11 | 11·08 | 15·9 | 6·08 | 8·2 : 12·0 | 2 |
| 56 | ... | 6 31·4 | 11 34 | 11·08 | 13·5 | 20·06 | 10·0 : 11·0 | 2 |
| 57 | ... | 6 32·4 | 11 36 | 11·08 | 211·3 | 6·38 | 11·5 : 12·0 | 2 |
| 58 | 11° 1245 | 6 33·1 | 11 39 | 11·08 | 131·5 | 16·44 | 9·2 : 11·5 | 2 |
| 59 | 40° 1751 | 6 50·4 | 40 43 | 7·17 | 7·5 | 24·98 | 8·3 : 11·5 | 5 |
| 60 | 40° 1753 | 6 50·6 | 40 40 | 7·17 | 47·1 | 24·45 | 7·5 : 11·5 | 5 |
| 61 | 40° 1756 | 6 51·2 | 40 47 | 7·17 | 345·0 | 23·70 | 9·3 : 9·5 | 5 |
| 62 | 3712 | 6 53·2 | 39 30 | 15·23 | 74·5 | 3·90 | 9·1 : 10·0 | 1 |
| 63 | ... | 6 57·0 | 29 46 | 15·24 | 193·0 | 10·14 | 11·5 : 11·7 | 2 |
| 64 | ... | 6 57·7 | 29 15 | 15·24 | 268·8 | 15·15 | 10·0 : 11·5 | 2 |
| 65 | ... | 6 58·1 | 29 2 | 10·00 | 282·5 | 13·50 | 10·0 : 10·2 | 6 |
| 66 | 4122 | 7 30·2 | 32 3 | 4·18 | 223·3 | 5·53 | 2·7 : 3·7 | 8 |
| 67 | ... | 7 55·6 | 28 51 | 7·23 | 31·9 | 9·35 | 10·8 : 11·2 | 3 |
| 68 | 4709 | 8 35·5 | 6 1 | 8·08 | 30·8 | 26·55 | 7·0 : 8·0 | 8 |
| 69 | 4972 | 9 9·7 | 53 0 | 7·08 | 67·0 | 19·12 | 7·4 : 7·4 | 12 |
| 70 | 5188 | 9 40·6 | 43 34 | 7·00 | 312·0 | 5·27 | 8·2 : 8·2 | 12 |
| 71 | 5523 | 10 37·8 | 47 1 | 7·50 | 136·5 | 9·19 | 8·2 : 9·7 | 12 |
| 72 | 5529 | 10 38·8 | 47 3 | 7·50 | 258·8 | 7·66 | 8·5 : 9·0 | 12 |

*List of Double Stars*—continued

| No. | Star ($\beta$.G.C. or B.D.) | R.A. (1930) | Dec. (1930) | Epoch 1900 + | Position Angle | Distance | Magnitudes | No. of Observations |
|---|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | | ° | ″ | | |
| 73 | 5537 | 10 39·5 | 46 34 | 7·50 | 87·7 | 287·73 | 5·2 : 7·2 | 12 |
| 74 | 48° 1963 | 11 40·2 | 47 40 | 6·90 | 260·6 | 4·92 | 9·4 : 11·5 | 9 |
| 75 | 6243 | 12 38·2 | −1 3 | 4·07 | 326·8 | 5·79 | 3·0 : 3·0 | 8 |
| 76 | ... | 13 14·6 | 35 57 | 7·36 | 72·1 | 4·06 | 10·5 : 11·0 | 3 |
| 77 | 6615 | 13 41·8 | 18 43 | 4·21 | 184·7 | 30·14 | 7·7 : 9·2 | 8 |
| 78 | 6869 | 14 22·5 | 23 58 | 4·11 | 73·8 | 45·41 | 9·0 : 9·1 | 8 |
| 79 | 7099 | 14 57·5 | 54 8 | 6·94 | 341·6 | 40·22 | 7·0 : 7·5 | 8 |
| 80 | 7162 | 15 9·7 | 19 32 | 4·27 | 8·9 | 24·23 | 6·1 : 7·0 | 8 |
| 81 | 39° 3028 | 16 39·9 | 39 15 | 4·25 | 281·9 | 4·90 | 9·4 : 11·5 | 4 |
| 82 | 7873 | 17 3·1 | 47 4 | 6·53 | 93·6 | 8·46 | 7·4 : 10·5 | 4 |
| 83 | 22° 3186 | 17 38·0 | 22 16 | 7·70 | 228·4 | 18·66 | 8·3 : 10·5 | 13 |
| | | | | 28·44 | 230·8 | 18·13 | ... | 4 |
| 84 | 8119 | 17 38·6 | 21 13 | 7·70 | 256·1 | 3·52 | 9·4 : 11·0 | 4 |
| 85 | 8125 | 17 38·9 | 21 27 | 7·70 | 355·4 | 7·96 | 9·4 : 9·6 | 14 |
| | | | | 28·44 | 355·5 | 8·06 | ... | 5 |
| 86 | 8157 | 17 43·0 | 21 54 | 7·70 | 139·1 | 24·96 | 7·5 : 8·5 | 14 |
| | | | | 28·44 | 139·2 | 24·92 | ... | 4 |
| 87 | 8795 | 18 43·4 | 16 54 | 6·70 | 37·5 | 4·96 | 8·3 : 10·7 | 10 |
| 88 | ... | 18 43·8 | 18 6 | 6·70 | 311·3 | 19·50 | 10·0 : 10·5 | 5 |
| 89 | 8812 | 18 44·7 | 11 3 | 6·80 | 159·0 | 106·24 | 7·0 : 7·7 | 12 |
| | | | | 22·01 | 158·5 | 105·13 | ... | 6 |
| | | | | 25·10 | 158·6 | 105·20 | ... | 8 |
| 90 | 8819 | 18 44·9 | 10 34 | 6·33 | 337·7 | 7·72 | 8·0 : 12·0 | 1 |
| | | | | 22·01 | 339·7 | 7·52 | ... | 7 |
| 91 | 10° 3665 | 18 45·2 | 10 41 | 6·37 | 324·9 | 32·96 | 7·7 : 10·5 | 8 |
| | | | | 22·01 | 328·0 | 39·20 | ... | 6 |
| | | | | 25·10 | 329·0 | 40·87 | ... | 5 |
| 92 | ... | 18 46·1 | 18 8 | 6·70 | 338·0 | 20·64 | 10·0 : 10·3 | 8 |
| 93 | ... | 19 1·3 | 7 49 | 7·71 | 280·3 | 2·98 | 11·0 : 11·5 | 4 |
| 94 | 9028 | 19 3·0 | 7 3 | 7·40 | 290·2 | 8·23 | 7·1 : 7·8 | 12 |
| 95 | ... | 19 7·8 | 49 55 | 26·71 | 45·8 | 9·49 | 11·0 : 11·5 | 2 |
| 96 | 9137 | 19 10·3 | 49 42 | 4·57 | 218·0 | 9·31 | 6·0 : 6·5 | 7 |
| | | | | 26·52 | 215·5 | 8·84 | ... | 15 |
| 97 | 49° 2961 | 19 10·9 | 49 38 | 4·55 | 210·1 | 34·26 | 8·8 : 9·8 | 8 |
| | | | | 26·73 | 210·4 | 33·25 | ... | 10 |
| 98 | 9397 | 19 30·0 | −0 25 | 7·83 | 107·2 | 6·05 | 8·5 : 12·0 | 4 |
| 99 | ... | 19 42·1 | 33 9 | 26·67 | 149·1 | 8·12 | 10·0 : 12·0 | 2 |
| 100 | ... | 19 43·0 | 33 27 | 26·67 | 270·0 | 14·34 | 8·5 : 12·0 | 6 |
| 101 | ... | 19 43·1 | 33 31 | 6·81 | 321·1 | 6·70 | 10·0 : 10·2 | 7 |
| | | | | 26·67 | 322·2 | 6·49 | ... | 6 |

*List of Double Stars*—continued

| No. | Star ($\beta$.G.C. or B.D.) | R.A. (1930) | Dec. (1930) | Epoch 1900 + | Position Angle | Distance | Magnitudes | No. of Observations |
|---|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | | ° | ″ | | |
| 102 | ... | 19 43·5 | 33 19 | 26·67 | 100·0 | 9·54 | 10·0 : 12·0 | 4 |
| 103 | 9617 | 19 43·8 | 33 34 | 26·67 | 70·6 | 25·86 | 5·1 : 8·1 | 6 |
| 104 | ... | 19 45·5 | 34 8 | 26·67 | 295·5 | 11·58 | 11·0 : 12·0 | 2 |
| 105 | ... | 19 45·7 | 34 4 | 26·67 | 330·5 | 14·36 | 10·2 : 12·0 | 2 |
| 106 | ... | 19 46·0 | 33 31 | 26·67 | 285·1 | 4·80 | 9·0 : 11·0 | 6 |
| 107 | ... | 19 46·5 | 33 30 | 6·81 | 199·0 | 15·40 | 10·0 : 10·2 | 8 |
| 108 | ... | 19 48·6 | 37 45 | 9·60 | 48·1 | 5·84 | 9·8 : 12·0 | 1 |
| 109 | ... | 19 48·7 | 37 14 | 9·60 | 51·8 | 7·61 | 10·2 : 11·0 | 5 |
| 110 | ... | 19 48·7 | 37 50 | 9·60 | 117·9 | 10·71 | 10·5 : 10·8 | 5 |
| 111 | 9699 | 19 49·5 | 37 51 | 9·60 | 166·4 | 11·22 | 10·0 : 10·0 | 7 |
| 112 | 37° 3657 | 19 50·3 | 37 13 | 9·60 | 108·2 | 14·42 | 8·8 : 11·5 | 5 |
| 113 | ... | 19 51·2 | 37 31 | 9·60 | 220·3 | 11·62 | 10·5 : 10·8 | 5 |
| 114 | 37° 3673 | 19 52·3 | 37 13 | 9·60 | 7·9 | 14·29 | 9·3 : 11·5 | 4 |
| 115 | 36° 3767 | 19 52·4 | 36 41 | 9·60 | 13·9 | 7·52 | 9·3 : 10·0 | 5 |
| 116 | ... | 19 52·8 | 37 30 | 9·60 | 253·5 | 10·83 | 9·2 : 10·2 | 5 |
| 117 | 37° 3679 | 19 53·3 | 37 52 | 9·60 | 356·6 | 11·48 | 9·5 : 10·8 | 4 |
| 118 | 22° 3919 | 20 2·8 | 23 1 | 8·12 | 106·2 | 11·85 | 10·0 : 11·5 | 7 |
| 119 | 33° 3832 | 20 13·3 | 33 50 | 9·60 | 191·5 | 19·90 | 9·2 : 10·0 | 6 |
| 120 | 33° 3870 | 20 18·4 | 34 11 | 9·60 | 53·8 | 9·07 | 8·7 : 10·7 | 5 |
| 121 | 4° 4501 | 20 33·5 | 4 37 | 7·78 | 120·5 | 18·76 | 9·5 : 11·0 | 8 |
| 122 | 60° 2150 | 20 39·7 | 61 1 | 26·70 | 354·2 | 14·47 | 9·5 : 12·0 | 2 |
| 123 | 10553 | 20 45·3 | 62 22 | 14·80 | 69·4 | 9·23 | 9·0 : 12·0 | 6 |
| 124 | ... | 20 49·6 | 61 38 | 26·0 | 153·8 | 15·58 | 10·0 : 11·5 | 4 |
| 125 | 10719 | 21 1·3 | 41 21 | 9·60 | 184·4 | 57·61 | 5·7 : 8·0 | 9 |
| 126 | 40° 4399 | 21 1·8 | 40 40 | 9·60 | 42·4 | 6·87 | 9·3 : 11·0 | 7 |
| 127 | ... | 21 2·6 | 44 27 | 10·68 | 68·4 | 11·34 | 11·0 : 11·5 | 5 |
| 128 | ... | 21 4·3 | 46 14 | 10·79 | 86·9 | 22·60 AB | 9·5 : 11·0 | 5 |
| | | | | | 195·5 | 15·21 AC | 9·5 : 11·5 | 3 |
| 129 | 10755 | 21 4·4 | 41 8 | 9·60 | 47·2 | 14·22 | 8·9 : 8·9 | 8 |
| 130 | ... | 21 5·5 | 44 48 | 10·68 | 8·4 | 8·44 | 11·0 : 11·5 | 4 |
| 131 | 10775 | 21 6·2 | 17 4 | 7·20 | 291·6 | 17·09 | 9·2 : 11·0 | 6 |
| | | | | 26·72 | 293·0 | 17·04 | ... | 12 |
| 132 | 46° 3199 | 21 6·4 | 46 29 | 10·79 | 131·3 | 17·60 | 9·5 : 9·8 | 7 |
| 133 | 45° 3430 | 21 7·3 | 46 7 | 10·79 | 170·0 | 12·68 | 8·0 : 11·0 | 7 |
| 134 | 10797 | 21 7·4 | 44 4 | 10·68 | 228·6 | 11·44 | 9·0 : 10·0 | 9 |
| 135 | 46° 3207 | 21 7·5 | 46 34 | 10·79 | 271·5 | 12·00 | 10·0 : 12·0 | 4 |
| 136 | ... | 21 7·6 | 46 23 | 10·79 | 148·6 | 8·50 | 10·0 : 11·5 | 2 |
| 137 | 45° 3431 | 21 7·7 | 45 46 | 10·79 | 97·6 | 8·47 | 9·5 : 11·0 | 5 |

*List of Double Stars*—continued

| No. | Star ($\beta$.G.C. or B.D.) | R.A. (1930) | Dec. (1930) | Epoch 1900 + | Position Angle | Distance | Magnitudes | No. of Observations |
|---|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | | ° | ″ | | |
| 138 | ... | 21 8·5 | 45 13 | 10·68 | 7·7 | 3·29 | 9·5 : 9·6 | 4 |
| 139 | 45° 3443 | 21 9·6 | 45 32 | 10·79 | 51·5 | 3·53 | 10·0 : 11·5 | 6 |
| 140 | ... | 21 54·8 | 44 31 | 9·68 | 23·1 | 10·41 | 10·0 : 11·5 | 1 |
| 141 | ... | 21 55·3 | 59 19 | 9·60 | 55·7 | 5·24 | 10·5 : 12·0 | 2 |
| | | | | 22·73 | 54·0 | 5·63 | ... | 3 |
| 142 | ... | 21 55·4 | 44 7 | 9·68 | 159·1 | 10·90 | 10·0 : 11·0 | 1 |
| 143 | 43° 4095 | 21 55·5 | 44 2 | 9·68 | 225·8 | 10·83 | 9·4 : 9·6 | 6 |
| 144 | 43° 4098 | 21 55·8 | 44 21 | 9·68 | 247·1 | 9·28 | 8·0 : 9·5 | 9 |
| 145 | 11425 | 21 56·6 | 64 14 | 7·86 | 126·2 | 19·48 | 9·5 : 11·0 | 7 |
| | | | | 30·65 | 126·4 | 18·92 | ... | 2 |
| 146 | ... | 21 57·7 | 43 35 | 9·68 | 232·2 | 8·60 | 10·5 : 12·0 | 1 |
| 147 | ... | 21 57·7 | 59 10 | 9·60 | 196·4 | 10·07 | 9·5 : 12·0 | 2 |
| | | | | 21·75 | 197·2 | 10·67 | ... | 5 |
| 148 | 13601 | 21 57·8 | 44 46 | 9·68 | 293·0 | 3·50 | 9·2 : 11·0 | 6 |
| 149 | ... | 21 58·4 | 58 12 | 9·60 | 304·3 | 3·96 | 9·8 : 11·8 | 3 |
| | | | | 21·75 | 303·5 | 3·96 | ... | 6 |
| 150 | ... | 21 58·5 | 64 24 | 7·86 | 81·9 | 20·28 | 10·0 : 11·0 | 7 |
| | | | | 30·65 | 82·7 | 20·66 | ... | 2 |
| 151 | ... | 21 59·1 | 58 19 | 9·60 | 344·9 | 13·78 | 11·0 : 12·0 | 2 |
| | | | | 21·75 | 344·2 | 13·61 | ... | 4 |
| 152 | ... | 22 0·8 | 59 26 | 9·60 | 100·6 | 9·32 | 10·8 : 11·8 | 2 |
| | | | | 21·75 | 99·7 | 9·26 | ... | 1 |
| 153 | 11483 | 22 1·7 | 64 17 | 7·86 | 280·2 | 7·01 | 4·7 : 6·5 | 6 |
| 154 | ... | 22 8·2 | 56 39 | 7·92 | 348·0 | 4·45 | 10·0 : 11·0 | 4 |
| | | | | 28·66 | 349·7 | 4·18 | ... | 3 |
| 155 | 55° 2704 | 22 11·1 | 56 15 | 7·92 | 228·6 | 15·43 | 9·2 : 12·0 | 4 |
| | | | | 28·66 | 228·6 | 15·63 | ... | 3 |
| 156 | 11764 | 22 25·9 | 58 4 | 7·19 | 8·4 | 25·18 | 8·5 : 9·3 | 8 |
| | | | | 24·76 | 6·9 | 25·14 | ... | 8 |
| 157 | 11772 | 22 26·6 | 58 4 | 7·19 | 191·1 | 40·96 | 3·0 : 5·3 | 14 |
| | | | | 24·76 | 190·4 | 40·86 | ... | 10 |
| 158 | 52° 3237 | 22 29·6 | 52 52 | 11·03 | 32·5 | 9·12 | 8·8 : 10·0 | 6 |
| 159 | ... | 22 30·7 | 51 51 | 11·03 | 242·3 | 3·74 | 10·5 : 12·5 | 2 |
| 160 | ... | 22 31·1 | 51 51 | 11·03 | 149·5 | 12·59 | 11·5 : 12·0 | 4 |
| 161 | 11830 | 22 31·7 | 57 51 | 7·19 | 114·7 | 3·71 | 9·3 : 9·7 | 12 |
| | | | | 24·76 | 115·0 | 3·52 | ... | 6 |
| 162 | 11836 | 22 32·1 | 52 37 | 11·03 | 267·0 | 7·38 | 8·7 : 10·0 | 7 |
| 163 | ... | 22 32·4 | 52 21 | 11·03 | 249·7 | 17·98 | 9·8 : 12·0 | 4 |
| 164 | ... | 22 32·6 | 51 44 | 44·03 | 71·2 | 24·15 | 10·5 : 11·5 | 5 |
| 165 | ... | 22 32·8 | 52 1 | 11·03 | 34·0 | 17·82 | 11·0 : 11·2 | 5 |

*List of Double Stars*—continued

| No. | Star ($\beta$.G.C. or B.D.) | R.A. (1930) | Dec. (1930) | Epoch 1900 + | Position Angle | Distance | Magnitudes | No. of Observations |
|---|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | | ° | ″ | | |
| 166 | ... | 22 33·5 | 51 48 | 11·03 | 358·4 | 9·51 | 11·2 : 11·4 | 5 |
| 167 | ... | 22 34·4 | 52 59 | 11·03 | 53·7 | 12·70 | 10·5 : 10·6 | 5 |
| 168 | ... | 22 35·9 | 52 22 | 11·03 | 16·1 | 16·90 | 11·0 : 12·0 | 4 |
| 169 | ... | 22 36·6 | 52 14 | 11·03 | 77·3 | 11·34 | 11·0 : 12·0 | 4 |
| 170 | ... | 22 37·9 | 52 37 | 11·03 | 179·5 | 16·75 | 10·0 : 11·0 | 3 |
| 171 | 11924 | 22 40·1 | 29 52 | 7·72 | 338·5 | 91·58 | 4·0 : 10·0 | 11 |
| | | | | 26·69 | 338·4 | 91·40 | ... | 6 |
| 172 | 29° 4755 | 22 42·5 | 29 34 | 26·69 | 310·4 | 9·00 | 10·0 : 11·5 | 4 |
| 173 | 29° 4764 | 22 43·7 | 30 14 | 7·92 | 337·4 | 4·24 | 9·4 : 11·5 | 3 |
| | | | | 26·69 | 334·2 | 4·06 | ... | 6 |
| 174 | ... | 23 1·2 | 31 0 | 10·82 | 189·4 | 3·69 | 10·0 : 10·1 | 5 |
| 175 | 12164 | 23 3·4 | 30 18 | 10·82 | 358·7 | 3·69 | 8·4 : 10·5 | 7 |
| 176 | 12176 | 23 4·7 | 31 4 | 10·82 | 55·1 | 17·23 | 7·0 : 11·0 | 2 |
| 177 | 12701 | 23 58·4 | 26 44 | 25·93 | 331·1 | 66·42 | 6·0 : 8·5 | 5 |
| | | | | 26·75 | 331·0 | 67·50 | ... | 4 |

---

## *The Determination of Dynamical Parallaxes of Double Stars.*

By W. S. Finsen.

1. In this paper I have followed the notation of Professor H. N. Russell * as follows :—

$d_1$ = dynamical parallax, obtained from the observed motion on the assumption that the mass of the system is equal to the Sun's mass.
$s$ = separation of the components of a double star in seconds of arc.
$\theta$ = position angle.
$w$ = projected relative velocity in seconds of arc per year.
$i$ = angle between the radius vector and the line of sight.
$j$ = ,, direction of motion in orbit and the line of sight.
$\gamma$ = ,, plane of orbit and tangent plane.
$\phi$ = ,, plane of orbit and the plane containing the radius vector and the line of sight.

$a$, $e$, $r$, $E$ are the semi-major axis, eccentricity, radius vector and eccentric angle in the true orbit ; $\mathcal{J}$ is the angle between the radius vector and the direction of motion in the orbit, and

$$k^3 = \sin i \sin^2 j(2 - r/a).$$

Professor E. Hertzsprung has pointed out to me, when discussing dynamical parallaxes, that considerable improvement would probably result if use were made of the angle $\psi$ between the radius vector and the direction

* *Astronomical Journal*, **38**, 89, 1928.

of motion in the apparent orbit. This angle can, of course, be determined from the observations.

Comstock's formula * is

$$d_1 = s\sqrt[3]{\frac{\left(\frac{d\theta}{dt}\right)^2}{4\pi^2}}\ \sqrt[3]{\frac{s}{a(1-e^2)\cos^2\gamma}}. \tag{1}$$

An alternative form is

$$d_1 = \sqrt[3]{\frac{c^2}{\pi^2 s}}\ \sqrt[3]{\frac{s}{a(1-e^2)\cos^2\gamma}}, \tag{2}$$

where $c$ is the areal constant for the apparent orbit, and we have the relation

$$c = \tfrac{1}{2}s^2\frac{d\theta}{dt}.$$

In the ellipse we have

$$(1-e^2) = (1+e\cos E)(1-e\cos E)\sin^2\mathcal{J} = \frac{r}{a}\left(2-\frac{r}{a}\right)\sin^2\mathcal{J}. \tag{3}$$

Also

$$\frac{s}{r} = \sin i\,; \tag{4}$$

$$\cos\gamma = \sin\phi\,\sin i\,; \tag{5}$$

$$c^2 = \tfrac{1}{4}s^2w^2\sin^2\psi. \tag{6}$$

On substituting (3), (4), (5) and (6) in (2) we find

$$d_1 = \sqrt[3]{\frac{sw^2\sin^2\psi}{4\pi^2}}\ \sqrt[3]{\frac{1}{\sin i\sin^2\phi\,\sin^2\mathcal{J}\left(2-\frac{r}{a}\right)}}. \tag{7}$$

This is still Comstock's formula (the separation into factors being the same), but in a new guise. The second factor is a function of $i$, $\phi$, $e$, $E$, since $\mathcal{J}$, $\frac{r}{a}$ are functions of $e$ and $E$. Now

$$\sin\psi = \frac{\sin\phi\,\sin\mathcal{J}}{\sin j} \tag{8}$$

and $j$ is a function of $i$, $\phi$, $\mathcal{J}$. The first factor is therefore also a function of $i$, $\phi$, $e$, $E$. That is, the two factors of (7) are not independent, and the required average value of the second factor will vary with $\psi$. This explains the weakness of Comstock's method ; in this method a mean is taken of quantities which vary enormously, whereas in fact different values should be used for different values of $\psi$.

Now $\psi$ is an observable quantity. In Comstock's formula, as given by equation (7), we can transfer $\sin^2\psi$ from the first radical to the second, and using (8) we obtain

$$d_1 = \sqrt[3]{\frac{sw^2}{4\pi^2}}\ \sqrt[3]{\frac{1}{\sin i\sin^2 j\left(2-\frac{r}{a}\right)}}, \tag{9}$$

* *Publications of the Washburn Observatory*, **12**, 31, 1908.

which is the Russell-Hertzsprung * formula. On performing the same operation on Comstock's formula as given by equation (2), after writing the second factor in the form of equation (7), we have

$$d_1 = \sqrt[3]{\frac{c^2}{\pi^2 s \sin^2 \psi}} \sqrt[3]{\frac{1}{\sin i \sin^2 j\left(2 - \frac{r}{a}\right)}}. \qquad (10)$$

We thus have the alternative forms (9) and (10) for the Russell-Hertzsprung formula ; in (10) full use is made of observed data, $c$ being found from the whole observed arc by one of several methods, $s$ and $\psi$ being the values for the middle of the arc, where both are best determined.

The second factor has to be averaged. It will still vary slightly with $\psi$, owing, among other things, to the fact that the frequency distribution of $e$ will vary with $\psi$. It may be worth while to take account of this.

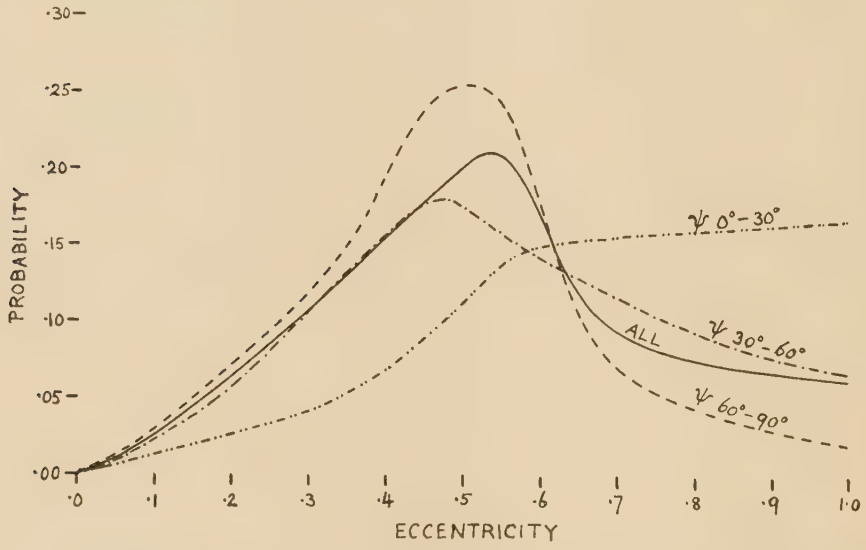
2. In order to see how $P(e)$, the probability that the eccentricity does not exceed $e$, changes with $\psi$, the values of $\psi$ at about the year 1900 were read off from plots of measures of 111 binaries with known orbits, once for each binary. The following table gives $P(e)$ for each of the three groups, $\psi = 0° - 30°$, $30° - 60°$, $60° - 90°$ ; also for all the 124 binaries now available (irrespective of $\psi$):—

| $\psi$ / $e$ | $P(e)$ 0°–30° | $P(e)$ 30°–60° | $P(e)$ 60°–90° | $P(e)$ all |
|---|---|---|---|---|
| ·0 | ·000 | ·000 | ·000 | ·000 |
| ·1 | ·006 | ·008 | ·016 | ·018 |
| ·2 | ·026 | ·049 | ·069 | ·066 |
| ·3 | ·062 | ·131 | ·165 | ·147 |
| ·4 | ·118 | ·268 | ·317 | ·274 |
| ·5 | ·207 | ·445 | ·548 | ·453 |
| ·6 | ·348 | ·607 | ·784 | ·651 |
| ·7 | ·505 | ·737 | ·887 | ·780 |
| ·8 | ·668 | ·843 | ·939 | ·863 |
| ·9 | ·834 | ·929 | ·975 | ·935 |
| 1·0 | 1·000 | 1·000 | 1·000 | 1·000 |
| $\bar{e^2}$ | ·501 | ·350 | ·264 | ·330 |
| $\bar{k}$ | ·663 | ·696 | ·707 | ·693 |
| $\bar{k}_{\text{emp.}}$ | ·722 | ·931 | ·783 | |
| | ± 18 per cent. p.e. | ± 22 per cent. p.e. | ± 26 per cent. p.e. | |

It will be noticed that $P(e)$ is very different for the four groups. The integrations were then completed for each $P(e)$ function, using the method

* *A.N.*, **190**, 113, 1911 ; *Science*, N.S., **34**, 524, 1911.

described by Professor Russell. The results show that changes in the $P(e)$ function have very little effect. There is evidently not much danger in assuming that an $e$-distribution derived from orbit pairs will be applicable to the slow-moving pairs with much longer average periods. These values of $\bar{k}$ cannot, however, be used as they stand, as no account is taken of the change in the $i$-distribution with different average values of $\psi$, an effect for which it would be difficult to allow. The different values of $\bar{k}$ as determined empirically are therefore given in the table; they include this effect, and may be of some use.




3. The derivation of Jackson's * formula directly from Comstock's is also instructive. We consider Comstock's formula in the form

$$d_1 = \sqrt[3]{\frac{c^2}{\pi^2 s}} \sqrt[3]{\frac{1}{\sin i \sin^2 \phi \sin^2 \mathfrak{f}\left(2 - \frac{r}{a}\right)}}$$

(from (2) and (7)).

We have the well-known formula in the theory of Keplerian motion,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta'}{dt}\right)^2 = -\frac{h^2}{lr^2},$$

where $\theta'$ is the position angle in the true orbit.

$\frac{h}{2}$ „ constant of areas „ „

$l$ „ semi-latus rectum.

Denoting

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta'}{dt}\right)^2$$

* *M.N.*, **81**, 5, 1920.

by $\alpha'$, we obtain $$h^2 = -lr^2\alpha'. \qquad (11)$$

Now

$$h^2 = \frac{4c^2}{\cos^2\gamma} = \frac{4c^2}{\sin^2\phi\sin^2 i}, \qquad r^2 = \frac{s^2}{\sin^2 i}, \qquad \alpha' = \frac{\alpha}{\sin i},$$

where $\alpha$ is the central acceleration in the apparent orbit. Also

$$l = \frac{b^2}{a} = a(1-e^2) = \frac{r(1-e^2)}{1-e\cos E} = \frac{s(1-e^2)}{(1-e\cos E)\sin i}.$$

On substituting these relations in (11),

$$c^2 = -\frac{\alpha(1-e^2)s^3\sin^2\phi}{4\sin^2 i(1-e\cos E)}. \qquad (12)$$

In the ellipse,

$$\sin^2\mathcal{J} = \frac{(1-e^2)}{1-e^2\cos^2 E} \quad \text{and} \quad \left(2-\frac{r}{a}\right) = 1+e\cos E.$$

$$\therefore \sin^2\mathcal{J}\left(2-\frac{r}{a}\right) = \frac{(1-e^2)}{1-e\cos E}. \qquad (13)$$

Substituting (12) and (13) in Comstock's formula,

$$d_1 = \sqrt[3]{-\frac{\alpha(1-e^2)s^2\sin^2\phi}{4\pi^2\sin^2 i(1-e\cos E)}}\ \sqrt[3]{\frac{1-e\cos E}{\sin i\sin^2\phi(1-e^2)}}.$$

This shows again the weakness of Comstock's formula: the two factor- are strongly correlated. Remembering that

$$\alpha = \frac{d^2s}{dt^2} - s\left(\frac{d\theta}{dt}\right)^2,$$

we obtain finally

$$d_1 = \sqrt[3]{\frac{s^3\left(\frac{d\theta}{dt}\right)^2 - s^2\frac{d^2s}{dt^2}}{4\pi^2}} \cdot \frac{1}{\sin i}, \qquad (14)$$

which is Jackson's formula.

4. There are several instances of binaries where, owing to lack of observations, only a small observed arc is available but the period is known.

For an orbit pair,

$$\begin{aligned} d_1 &= aP^{-\frac{2}{3}} \\ &= \frac{a}{r}\cdot\frac{r}{1}\cdot P^{-\frac{2}{3}} \\ &= sP^{-\frac{2}{3}}\left(\frac{a}{r}\cdot\frac{1}{\sin i}\right). \end{aligned} \qquad (15)$$

We require to know the value of

$$\overline{\left(\frac{r}{a}\cdot\sin i\right)}^{-1}\ *.$$

* *Astronomical Journal*, **38**, 90, formula (10), 1928.

$\frac{r}{a}$ and sin $i$ are independent and

$$\overline{\sin i} = \frac{\pi}{4},$$

$$\left(\overline{\frac{r}{a}}\right) = \frac{\int_0^\pi (1 - e\cos E)^2 dE}{\int_0^\pi (1 - e\cos E) dE} = 1 + \tfrac{1}{2}\overline{e^2}.$$

The value of $\overline{e^2}$ found earlier is 0·330. We therefore obtain for the hypothetical parallax the expression

$$h_1 = 1{\cdot}09 s P^{-\frac{2}{3}}.$$

If we compare Jackson's formula (14) with formula (15) we readily obtain

$$P = \frac{2\pi}{\left[\left(\overline{\frac{d\theta}{dt}}\right)^2 - \overline{\frac{1}{s}\frac{d^2 s}{dt^2}}\right]^{\frac{1}{2}}} \cdot \left(\overline{\frac{a}{r}}\right)^{\frac{3}{2}}. \tag{16}$$

In order that this formula may be of use for deriving the " hypothetical period " of a slow-moving physical pair, an average value of the second factor must be substituted. The logarithm of the period is generally the most useful form :

$$\log_{10} P = \log_{10} 2\pi - \tfrac{1}{2}\log_{10}\left[\left(\overline{\frac{d\theta}{dt}}\right)^2 - \overline{\frac{1}{s}\frac{d^2 s}{dt^2}}\right] + \tfrac{3}{2}\log_{10}\overline{\frac{a}{r}};$$

or since

$$\frac{r}{a} = 1 - e\cos E,$$

$$\log_{10} P = \log_{10} 2\pi - \tfrac{1}{2}\log_{10}\left[\left(\overline{\frac{d\theta}{dt}}\right)^2 - \overline{\frac{1}{s}\frac{d^2 s}{dt^2}}\right] - \tfrac{3}{2}\log_{10}(1 - e\cos E);$$

$$\overline{\log_{10}(1 - e\cos E)} = \frac{M}{\pi}\int_0^\pi (1 - e\cos E)\log(1 - e\cos E) dE. \quad (M = 0{\cdot}4343).$$

Expanding and integrating the resulting series,

$$\overline{\log_{10}(1 - e\cos E)} = M\left(\frac{e^2}{4} + \frac{e^4}{32} + \frac{e^6}{96} + \ldots + \frac{(2n-2)!}{2^{2n}(n!)^2}e^{2n} + \cdots\right);$$

$$\overline{e^2} = 0{\cdot}330;$$

$$\overline{e^4} = 0{\cdot}169;$$

$$\overline{e^6} = 0{\cdot}109.$$

Therefore

$$\overline{\log_{10}(1 - e\cos E)} = 0{\cdot}0386,$$

with an error less than 0·0017.

Finally,

$$\log_{10} P = 0{\cdot}740 - \tfrac{1}{2}\log_{10}\left[\left(\overline{\frac{d\theta}{dt}}\right)^2 - \overline{\frac{1}{s}\frac{d^2 s}{dt^2}}\right]. \tag{17}$$

*Union Observatory, Johannesburg :*
*1931 September 7.*

## *Stellar Parallaxes determined at the Royal Observatory, Greenwich.*

(*Communicated by the Astronomer Royal.*)

The following list is a continuation of the results obtained with the 26-inch telescope and published annually in the November number of the *Monthly Notices*.

In the column " Proper Motion " the first value given is from meridian observations ; the second is the relative proper motion derived from the photographs. The motions in both R.A. and Dec. are expressed in arc of a great circle.

In calculating the parallax the relative proper motion has been used, *not* the meridian proper motion.

In computing the P.E. an average value $\pm$ ″·025 has been adopted as the probable error of a single plate of unit weight. This value has been derived from a discussion of the residuals of 333 determinations of parallax, each of which depends on 23 plates on the average. The P.E. has accordingly been calculated by the formula

$$r = \frac{\pm\,''\cdot 025}{\sqrt{w}}.$$

The first P.E. given is that of the parallax, the second is the P.E. of the relative proper motion.

Generally, six comparison stars of mean magnitude 11–12 have been used, of which good images are obtained with an exposure of three minutes. The magnitude of the parallax star has been reduced to equality by means of the rotating sector and (for the brighter stars) local desensitisation by means of a weak solution of copper sulphate.

The visual magnitude is taken from the *Bonn Durchmusterung*, the photographic from the *Greenwich Astrographic Catalogue*.

| No. | Name and B.D. No. | Mag. Vis. | Mag. Photog. | Spectral Type | R.A. 1900·0 | Dec. 1900·0 | Proper Motion R.A. | Proper Motion Dec. | Parallax | P.E. | No. of Plates |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | h m s | | ″ | ″ | ″ | ″ | |
| 1 | 75° 58 | | 7·1 | K0 | 1 13 48 | +76° 11′ | −·031 | −·069 | +·039 | ±·007 | 25 |
| | | | 7·8 | | | | −·018 | | | ±·007 | |
| 2 | Br. 396 | | 5·5 | A2 | 2 56 11 | +81° 5′ | −·069 | −·011 | +·014 | ±·007 | 29 |
| | 80° 97 | | 6·1 | | | | −·053 | | | ±·007 | |
| 3 | L. 8441 | | 6·3 | F5 | 4 32 8 | +76° 25′ | +·084 | −·149 | +·009 | ±·005 | 28 |
| | 76° 174 | | 6·7 | | | | +·059 | | | ±·006 | |
| 4 | Gr. 861 | | 7·0 | G5 | 4 39 32 | +75° 32′ | +·098 | −·168 | +·008 | ±·006 | 29 |
| | 75° 193 | | 7·5 | | | | +·064 | | | ±·007 | |

| No. | Name and B.D. No. | Mag. Vis. Photog. | Spectral Type | R.A. Dec. 1900·0 | Proper Motion R.A. | Proper Motion Dec. | Parallax | P.E. | No. of Plates |
|---|---|---|---|---|---|---|---|---|---|
| | | | | h m s | ″ | ″ | ″ | ″ | |
| 5 | Gr. 896 | 7·0 | F5 | 4 51 20 | +·080 | −·086 | +·019 | ±·005 | 31 |
| | 68° 357 | 7·3 | | +69° 1′ | +·095 | | | ±·003 | |
| 6 | O.A. 5424 | 6·5 | K0 | 4 58 40 | +·067 | −·067 | +·003 | ±·006 | 27 |
| | 69° 302 | 7·3 | | +69° 30′ | +·064 | | | ±·003 | |
| 7 | O.A. 5856 | 7·0 | F0 | 5 22 28 | −·033 | −·164 | +·036 | ±·006 | 26 |
| | 67° 390 | 6·8 | | +67° 56′ | +·017 | | | ±·006 | |
| 8 | 73° 300 | 9·4 | | 5 39 2 | +·016 | −·160 | +·028 | ±·005 | 32 |
| | | 9·8 | | +73° 49′ | +·015 | | | ±·005 | |
| 9 | Anon. | ... | | 6 20 31 | −·039 | −·208 | +·018 | ±·005 | 33 |
| | | 12·5 | | +64° 37′ | −·050 | | | ±·004 | |
| 10 | L.F. 228 | 6·0 | F2 | 6 25 18 | −·149 | −·032 | +·034 | ±·005 | 31 |
| | 73° 340 | 6·3 | | +73° 46′ | −·135 | | | ±·005 | |
| 11 | A.C. 69° 3479 | ... | | 8 4 2 | +·013 | −·305 | −·002 | ±·006 | 27 |
| | | 10·6 | | +69° 14′ | +·026 | | | ±·004 | |
| 12 | 70° 510 | 7·3 | G5 | 8 16 10 | −·081 | +·041 | +·012 | ±·006 | 29 |
| | | 8·0 | | +70° 27′ | −·090 | | | ±·005 | |
| 13 | 23 UMa | 3·8 | F0 | 9 23 39 | +·114 | +·025 | +·042 | ±·007 | 22 |
| | 63° 845 | 4·0 | | +63° 30′ | +·105 | | | ±·004 | |
| 14 | L.F. 2309 | 7·5 | G5 | 13 26 43 | +·035 | +·109 | +·009 | ±·008 | 26 |
| | 84° 311 | 7·7 | | +83° 49′ | +·022 | | | ±·006 | |
| 15 | L.F. 2506 | 7·3 | F8 | 14 36 53 | −·131 | −·026 | +·049 | ±·008 | 26 |
| | 64° 1017 | 7·8 | | +64° 43′ | −·124 | | | ±·005 | |
| 16 | L. 29965 | 9·0 | G5 | 16 18 13 | +·015 | −·106 | +·020 | ±·008 | 30 |
| | 67° 937 | 9·3 | | +67° 30′ | +·014 | | | ±·006 | |
| 17 | 76° 614 | 9·4 | | 16 37 13 | +·149 | +·112 | +·048 | ±·009 | 23 |
| | | 10·5 | | +76° 10′ | +·089 | | | ±·007 | |
| 18 | L.F. 2864 | 7·8 | G5 | 16 58 36 | −·198 | +·088 | +·025 | ±·009 | 28 |
| | 71° 817 | 8·3 | | +71° 37′ | −·198 | | | ±·007 | |
| 19 | 79° 537 | 9·3 | | 17 23 31 | −·121 | −·138 | −·008 | ±·009 | 30 |
| | | 10·4 | | +79° 40′ | −·077 | | | ±·004 | |
| 20 | L. 33223 | 6·7 | K0 | 17 56 4 | +·026 | +·009 | −·017 | ±·010 | 27 |
| | 69° 958 | 7·4 | | +69° 38′ | +·019 | | | ±·006 | |
| 21 | L. 33826 | 7·1 | K0 | 18 11 44 | +·019 | +·031 | −·003 | ±·009 | 29 |
| | 66° 1087 | 8·2 | | +66° 6′ | −·028 | | | ±·007 | |

| No. | Name and B.D. No. | Mag. Vis. Photog. | Spectral Type | R.A. Dec. 1900·0 | Proper Motion R.A. | Proper Motion Dec. | Parallax | P.E. | No. of Plates |
|---|---|---|---|---|---|---|---|---|---|
| | | | | h m s | ″ | ″ | ″ | ″ | |
| 22 | 50 Dra | 5·8 | Ao | 18 49 36 | −·013 | +·077 | −·004 | ±·009 | 26 |
| | 75° 682 | 5·2 | | +75° 19′ | −·012 | | | ±·007 | |
| 23 | δ Dra | 3·4 | Ko | 19 12 32 | +·099 | +·089 | +·021 | ±·009 | 27 |
| | 67° 1129 | 4·3 | | +67° 29′ | +·084 | | | ±·007 | |
| 24 | R. 4476 | 8·4 | Ko | 19 33 54 | +·047 | +·078 | −·006 | ±·010 | 24 |
| | 85° 329 | 8·8 | | +85° 53′ | +·033 | | | ±·007 | |
| 25 | L. 39216 | 8·3 | Ko | 20 8 23 | −·007 | −·060 | +·003 | ±·008 | 32 |
| | 80° 644 | 8·9 | | +80° 24′ | −·009 | | | ±·004 | |
| 26 | A.C. 65° 6955 | ... | | 20 29 0 | +·449 | +·283 | +·153 | ±·008 | 25 |
| | | 11·5 | | +65° 6′ | +·443 | | | ±·007 | |
| 27 | Br. 2749 | 5·3 | Ko | 20 52 8 | −·026 | −·030 | +·007 | ±·008 | 29 |
| | 80° 672 | 6·3 | | +80° 11′ | −·014 | | | ±·006 | |
| 28 | Gr. 3409 | 6·0 | F2 | 21 5 48 | −·058 | −·128 | +·029 | ±·008 | 24 |
| | 70° 1164 | 6·1 | | +71° 2′ | −·039 | | | ±·007 | |
| 29 | L.F. 3995–6 | 7·2 | A5 | 21 50 24 | +·095 | +·013 | +·030 | ±·008 | 25 |
| | 83° 618 | 7·3 | | +83° 34′ | +·115 | | | ±·007 | |
| 30 | δ Cep | 3·6–4·1 | F2–G2 | 22 25 27 | +·011 | +·003 | +·009 | ±·008 | 25 |
| | 57° 2548 | 3·9–4·5 | | +57° 54′ | +·032 | | | ±·007 | |
| 31 | o Cep | 5·2 | G5 | 23 14 31 | +·081 | +·015 | +·018 | ±·008 | 30 |
| | 67° 1514 | 5·6 | | +67° 34′ | +·042 | | | ±·004 | |
| 32 | 68° 1405 | 9·4 | | 23 47 30 | +·218 | +·006 | +·025 | ±·008 | 25 |
| | | 10·3 | | +68° 56′ | +·229 | | | ±·007 | |

---

## *The Gravitational Effect of Radiation on Stellar Structure.*

By G. C. McVittie, Ph.D.

### I. *Introduction*

In the theory of stellar structure lately put forward by E. A. Milne,* the temperatures in the interiors of the models have been shown to be so large that the radiation-density corresponding to them is comparable with the gas-density. In the innermost parts of these models, therefore, the gravitational effect of the radiation must also be comparable with that of

* E. A. Milne, *M.N.*, **91**, 4–55, 1930 November.

the matter. The object of the present paper is to derive the equation of mechanical equilibrium appropriate to a region containing matter and radiation, and to apply the resulting new equation to the study of the central regions of Milne's " standard " models.

In § II the new equation of mechanical equilibrium is derived as an approximation to the exact equations of general relativity for a region containing matter and radiation of comparable densities. The approximation is made with respect to the constant of gravitation, $G$, and the new equation is shown to hold only if powers of $G/c^4$ ($c$ = velocity of light) higher than the first are neglected.

The attempt to take into account, by the use of this new equation, the weight of the radiation in the perfect-gas or degenerate-gas regions of Milne's " standard " models has led to rather intractable equations, and not much progress has been made in this direction. There is one case, however, in which the new equations reduce to Emden's equation for $n = 1$ : this occurs when we consider a region of incompressible matter and radiation of high density. From § III onwards, therefore, we give the theory of models consisting, first, of an outer envelope of perfect gas ; secondly, of an inner zone of degenerate gas ; and thirdly, of a core of incompressible matter and radiation. In the first and second regions the gravitational effect of the radiation is neglected. The rate of energy generation and the value of the opacity coefficient, $k$, are both assumed to be constant throughout the whole model.

In the perfect-gas region, it is known that the structure is determined by Emden's equation :

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

with $n = 3$, whereas in the degenerate-gas region the index is $n = 3/2$. In all cases solutions of Emden's equation are of three types :

1. Emden solutions, with $\theta$ finite at $\xi = 0$.

2. Centrally condensed solutions, with $\theta \to +\infty$ at $\xi = 0$.

3. Collapsed solutions, in which $\theta = 0$ for some finite value of $\xi$ and $\theta$ increases to a maximum, then diminishes.

In the core, the structure depends on Emden's equation $n = 1$, the Emden solution of which is $\theta = a \sin \xi/\xi$ ($a$ = constant).

We establish the equations of fit between the degenerate-gas zone and the core. We have then to show that it is possible to solve these equations for an Emden solution in the core and solutions of type (1), (2) or (3) in the degenerate-gas zone. We show that in case (3) the equations of fit always possess a solution, but in cases (1) and (2), although they do so in general, yet there are exceptions in which no solution is possible. In case (2) the exceptional models are those with very large masses combined with very small luminosities. Whenever a solution does exist, however, the gas density is continuous at the interface between the degenerate gas and the incompressible core.

The models with solutions of type (1) or (3) in the degenerate-gas zone

are not discussed in detail. The present treatment does not give results differing sufficiently widely from those found by Milne * in his theory of white dwarfs to be of interest. The temperature structure in the core is different from Milne's, and it is possible that a core should exist even when the solution in the degenerate-gas zone is of type (1); otherwise the conclusions are the same.

The case when the solution in the degenerate gas is of type (2) is gone into in detail. For in this case, in Milne's theory, it was necessary to assume a discontinuity of density † at the interface with the solid core to obtain a finite value of the density at the centre of the model. Alternatively, the core was assumed to be absent, but then the opacities of the perfect and degenerate gas had to be different. It is now shown that when the gravitational effect of the radiation due to the high central temperatures of these models is taken into account, the model can, in general, be finished off by a core of the type considered. In conclusion the theory is applied to actual giant stars, and it is shown that if the star is only slightly centrally condensed the core does exist. If, however, the star is very highly centrally condensed the existence of the core is doubtful, and this point can only be definitely settled when tables of centrally condensed solutions of Emden's equations for $n=3$ and $n=3/2$ have been calculated.

## II. *The Equation of Mechanical Equilibrium*

We consider a mixture of matter and radiation having spherical symmetry, the density of the radiation being comparable with that of the matter. We suppose that the rate of flow of the radiation away from the central point is very slow compared with the velocity of light. In these circumstances, it is legitimate to assume that the line element of the space occupied by the distribution is of the form

$$ds^2 = e^{\nu}dt^2 - \frac{1}{c^2}(e^{\lambda}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2), \tag{1}$$

where $\nu$, $\lambda$ are functions of $r$ alone; and that the energy tensor is of the form

$$\left.\begin{aligned} T_4{}^4 &= \rho + u. \\ T_1{}^1 = T_2{}^2 &= T_3{}^3 = -(p+\bar{p}). \end{aligned}\right\} \tag{2}$$

In (2), $\rho$ is the density in ergs/cm.$^3$ of the matter; $u$, that of the radiation in ergs/cm.$^3$; and $p$ and $\bar{p}$ the gas- and radiation-pressures respectively.

The equations of gravitational equilibrium according to the general relativity theory are

$$R_{\nu}{}^{\mu} - \tfrac{1}{2}\delta_{\nu}{}^{\mu}R = -\kappa c T_{\nu}{}^{\mu}. \tag{3}$$

$R_{\nu}{}^{\mu}$ is the mixed Riemann-Christoffel tensor; $\delta_{\nu}{}^{\mu}$, the Kroenecker delta; $R$, the scalar curvature. The constant $\kappa$ is small and is known to be

* E. A. Milne, *loc. cit.* Sections (C) and (D).

† E. A. Milne, *M.N.*, **91**, 479–482, 1931 March.

proportional to the gravitational constant $G$. The exact relation between $\kappa$ and $G$ is found later.

Evaluating (3) for (1) and (2) we get the following equations :—

$$e^{-\lambda}\left(-\frac{\lambda'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=-\frac{\kappa}{c}(\rho+u)\,; \tag{4}$$

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=+\frac{\kappa}{c}(p+\bar{p})\,; \tag{5}$$

$$e^{-\lambda}\left(\tfrac{1}{2}\nu''-\tfrac{1}{4}\lambda'\nu'+\tfrac{1}{4}\nu'^{1\cdot 2}+\tfrac{1}{2}\frac{\nu'}{r}-\tfrac{1}{2}\frac{\lambda'}{r}\right)=\frac{\kappa}{c}(p+\bar{p}). \tag{6}$$

Here a dash denotes differentiation with respect to $r$.

Our object now is to obtain a differential equation connecting $p$, $\bar{p}$ and $\rho$, $u$ by the elimination of $\nu$ and $\lambda$ from equations (4), (5), (6). We do not attempt to perform this elimination rigorously : instead we assume that

$$\nu=1-\frac{\kappa}{c}V-\frac{\kappa^2}{c^2}U, \tag{7}$$

where $V$, $U$ are functions of $r$, and neglect all terms in $\nu$ involving powers of $\frac{\kappa}{c}$ higher than the second.

Before proceeding with the actual elimination we put the equations (4) to (6) into a more convenient form. We have that (4) can be integrated immediately, giving

$$e^{-\lambda}=1-\frac{\kappa}{c}\frac{M(r)}{r}. \tag{8}$$

Here we have put

$$M(r)=\int_0^r r^2(\rho+u)dr=\frac{c^2}{4\pi}\bar{M}(r), \tag{9}$$

where $\bar{M}(r)=\int_0^r 4\pi\frac{(\rho+u)}{c^2}dr$, and is therefore the mass in grammes inside $r$. From (9) we see that $M(r)$ is the mass inside $r$ expressed in ergs, except for a factor $\frac{1}{4\pi}$. We shall henceforward speak of $M(r)$ as the " mass inside $r$ in ergs."

Write

$$\omega=r-\frac{\kappa}{c}M(r). \tag{10}$$

We can then eliminate $(p+\bar{p})$ from (5) and (6), and by means of (8) and (10) reduce the resulting equation to

$$\nu''+\frac{\nu'}{2}\left(\frac{\omega'}{\omega}-\frac{3}{r}\right)+\tfrac{1}{2}\nu'^{1\cdot 2}+\frac{r\omega'-3\omega+2r}{\omega r^2}=0. \tag{11}$$

In this equation, substitute for $\nu$ from (7) and for $\omega$, $\omega'$ from (10) and

expand up to the second power of $\frac{\kappa}{c}$. Then, equating coefficients of successive powers of $\frac{\kappa}{c}$ to zero, we get the following equation for $U$ and $V$ :—

$$-V''+\frac{V'}{r}+\frac{3M(r)}{r^2}-(\rho+u)=0, \tag{12}$$

and

$$-U''+\frac{U'}{r}+\tfrac{1}{2}V'\left\{r(\rho+u)-\frac{M(r)}{r^2}\right\}+\tfrac{1}{2}V'^{1.2}+\frac{3M^2(r)}{r^4}-(\rho+u)\frac{M(r)}{r}=0. \tag{13}$$

A solution of (12) is

$$V'=-\frac{M(r)}{r^2}. \tag{14}$$

Hence (13) becomes

$$-U''+\frac{U'}{r}-\frac{3}{2}(\rho+u)\frac{M(r)}{r}+4\frac{M^2(r)}{r^4}=0. \tag{15}$$

We can now obtain the value of $(p+\bar{p})$ from (5). For, by (8) and (7), the equation (5) becomes

$$\frac{\kappa}{c}(p+\bar{p})=\left(1-\frac{\kappa}{c}\frac{M(r)}{r}\right)\left(\frac{1}{r^2}-\frac{\kappa}{c}\frac{V'}{r}-\frac{\kappa^2}{c^2}\frac{U'}{r}\right)-\frac{1}{r^2}.$$

Hence

$$p+\bar{p}=-\frac{\kappa}{c}\left(\frac{U'}{r}-\frac{M(r)V'}{r^2}\right).$$

Differentiate this equation once with respect to $r$ and substitute from (14) and (15) : we obtain

$$p'+\bar{p}'=-\tfrac{1}{2}\frac{\kappa}{c}\frac{M(r)}{r^2}(\rho+u). \tag{16}$$

This is the equation of mechanical equilibrium we wished to find. It is not rigorous, but holds only if powers of $\frac{\kappa}{c}$ higher than the second are neglected.

To obtain $\kappa$ in terms of $G$, replace $M(r)$ in (16) by $\bar{M}(r)$ from (9) and also $(\rho+u)$ by the density in grammes $\left(\frac{\rho+u}{c^2}\right)$. Then (16) must reduce to the ordinary equation of mechanical equilibrium,

$$p'+\bar{p}'=-G\frac{\bar{M}(r)}{r^2}\left(\frac{\rho+u}{c^2}\right).$$

Hence

$$G=\frac{\kappa c^3}{8\pi}$$

or

$$\frac{\kappa}{c}=\frac{8\pi G}{c^4}=2\times 10^{-48}\text{ C.G.S. units.} \tag{16.1}$$

### III. *The Incompressible Core*

We now apply equation (16) to a region in which the matter-density is constant whilst the radiation-density may vary. Such a region, we assume, may occur near the centre of a model which has an outer envelope of perfect gas and an inner zone of degenerate gas. The opacity, $k$, throughout the whole model is constant, as well as the rate of energy generation. In this paragraph we obtain the solution in the core. In the following paragraphs we show that composite models with such a core are possible.

Let $\rho_0$ be the density (in ergs) of the matter in the core. Then we have the equation (16),

$$\frac{dp}{dr}+\frac{d\bar{p}}{dr}=-\frac{\kappa}{2c}\frac{M(r)}{r^2}(\rho_0+u). \tag{16}$$

Since we can suppose the core to be in local thermodynamic equilibrium, the radiation-flux equation will have the usual form,

$$\frac{d\bar{p}}{dr}=-\frac{kL(r)}{4\pi c^3 r^2}\rho_0. \tag{17}$$

Let $M_0(r)$ denote the mass of the energy in the form of matter inside radius $r$ ; $M_1(r)$ the mass of the energy in the form of radiation. Then

$$M(r)=M_0(r)+M_1(r)$$

is the total mass within $r$.

We now define a constant, $\epsilon$, by means of the relation

$$\frac{4\pi\epsilon}{c^2}=\frac{L(r)}{M(r)}=\frac{L}{M},$$

where $L(r)$ is the luminosity of a sphere of radius $r$ and $L$, $M$ are the total luminosity and mass of the model. We call $\epsilon$ the " rate of energy generation," although in a region of mixed matter and radiation it is more properly termed the " rate of liberation of energy per erg in the sphere of radius $r$." For, from its definition, $4\pi\epsilon/c^2$ is the fraction of the energies $M_0(r)$ and $M_1(r)$ radiated away in one second. As far as the portion $4\pi\epsilon M_0/c^2$ of this energy, $L(r)$, is concerned, there is actually a transformation of energy from the form of matter to that of radiation. Hence, here $\epsilon$ is properly described as an " energy generation " coefficient. But the portion $4\pi\epsilon M_1/c^2$ of $L(r)$ merely leaks away with a velocity small compared with that of light. Hence, here $\epsilon$ is to be compared to the " coefficient of disintegration " of a radioactive substance, *i.e.* to the fraction of the mass lost by the emission of *material* particles ($\alpha$-rays).

The equation (17) now becomes

$$\frac{d\bar{p}}{dr}=-\frac{kL}{4\pi c^3 M}\frac{M(r)}{r^2}\rho_0=-\frac{\kappa}{2c}(1-\beta)\frac{M(r)}{r^2}\rho_0, \tag{18}$$

where

$$\beta=1-\frac{kL}{2\pi c^2\kappa M}.$$

Also we have

$$\frac{dM(r)}{dr} = (\rho_0 + u)r^2. \tag{19}$$

The gas-density is independent of $r$, but the gas-pressure may, of course, vary with $r$.* Also, since our equations are only first approximations to the equations of general relativity, we can still put

$$\bar{p} = \tfrac{1}{3}u.$$

Divide (17) by (18). We get

$$\frac{dp}{d\bar{p}} + 1 = \frac{\rho_0 + u}{\rho_0(1-\beta)},$$

the integral of which is

$$p = \frac{1}{6(1-\beta)\rho_0}\{(\beta\rho_0 + u)^2 - (\beta\rho_0 + u_0)^2\}, \tag{20}$$

where the constant of integration is adjusted so that

$$p = 0 \quad \text{when} \quad u = u_0.$$

Again making use of (18) and (19), we get

$$\frac{d}{dr}\left(\frac{r^2}{3}\frac{du}{dr}\right) = -(1-\beta)\frac{\kappa}{2c}\rho_0(\rho + u)r^2. \tag{21}$$

In this put

$$y = u + \rho_0 \tag{22}$$

and

$$r = \left(\frac{2c}{3\kappa(1-\beta)\rho_0}\right)^{\frac{1}{2}}\xi, \tag{23}$$

and (21) becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dy}{d\xi}\right) = -y. \tag{24}$$

This is the equation giving the variation of the radiation-density with distance from the centre. It is Emden's equation for $n = 1$, and accordingly its Emden solution is

$$y = A\frac{\sin\xi}{\xi}, \tag{25}$$

where $A$ is the value of $y$ at $\xi = 0$.

Equations (20), (22), (25) completely determine the state of the matter and the radiation in the core, when $A$, $u_0$ and $\rho_0$ are known.

From (18) the mass inside radius $r$ is

$$M(r) = -\left(\frac{2c}{3\kappa(1-\beta)\rho_0}\right)^{\frac{3}{2}}\xi^2\frac{dy}{d\xi}. \tag{26}$$

---

* See, *e.g.*, A. S. Eddington: *The Mathematical Theory of Relativity* (1924), § 72.

## IV. *The Equations of Fit*

Outside the core there is the zone of degenerate gas in which the gravitational effect of the radiation is neglected, so that the equations corresponding to (16), (18), (19) are now

$$\frac{dp}{dr}+\frac{d\bar{p}}{dr}=-\frac{\kappa}{2c}\frac{M(r)}{r^2}\rho\,;$$

$$\frac{d\bar{p}}{dr}=-\frac{\kappa}{2c}(1-\beta)\frac{M(r)}{r^2}\rho\,;$$

$$\frac{dM(r)}{dr}=\rho r^2\,;$$

with now the equation of state

$$p=\frac{K}{c^{\frac{10}{3}}}\rho^{\frac{5}{3}}=a\rho^{\frac{5}{3}}, \tag{27}$$

where $K$ is the degenerate-gas constant and $c$ the velocity of light, so that

$$a=3\cdot17\times10^{12}/3^{\frac{10}{3}}\times10^{\frac{10}{3}}=3\cdot8\times10^{-23}.$$

The solution in the degenerate-gas zone has been given by Milne.* In our notation it becomes

$$\left.\begin{aligned} &r=\left(\frac{5ca}{\kappa\beta}\lambda^{-\frac{1}{3}}\right)^{\frac{1}{2}}\eta\,; \qquad u=aT^4=3a\frac{(1-\beta)}{\beta}\lambda^{\frac{5}{3}}\psi^{\frac{5}{2}};\\ &\rho=\lambda\psi^{\frac{3}{2}}\,; \qquad M(r)=-\left(\frac{5c}{\kappa}\frac{a}{\beta}\right)^{\frac{3}{2}}\lambda^{\frac{1}{2}}\eta^2\frac{d\psi}{d\eta}\,;\\ &p=a\lambda^{\frac{5}{3}}\psi^{\frac{5}{2}}\,; \end{aligned}\right\} \tag{28}$$

where $\lambda$ is an arbitrary constant and $\psi$ and $\eta$ are connected by Emden's $n=\frac{3}{2}$ equation, viz.

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right)=-\psi^{\frac{3}{2}}. \tag{29}$$

At the interface between the core and the degenerate gas the values of the radius, radiation-density, matter-density, gas-pressure and the " mass inside radius $r$," obtained from the solution for the core must be equal, respectively, to the corresponding quantities obtained from the solution (28) for the degenerate-gas zone. If, then, $\psi_1$ and $\eta_1$ are the values of $\psi$ and $\eta$ at the interface, and $y_1$, $\xi_1$ the values of $y$, $\xi$, we obtain the equations

$$\left(\frac{5ca}{\kappa\beta}\lambda^{-\frac{1}{3}}\right)^{\frac{1}{2}}\eta_1=\left(\frac{2c}{3\kappa(1-\beta)\rho_1}\right)^{\frac{1}{2}}\xi_1\,; \tag{30}$$

$$3a\frac{(1-\beta)}{\beta}\lambda^{\frac{5}{3}}\psi_1^{\frac{5}{2}}=y_1-\rho_1\,; \tag{31}$$

$$\lambda\psi_1^{\frac{3}{2}}=\rho_1\,; \tag{32}$$

* E. A. Milne, *M.N.*, **91**, 4, § 19, 1930.

$$\left(\frac{5c}{\kappa}\frac{\alpha}{\beta}\right)^{\frac{3}{2}}\lambda^{\frac{1}{2}}\eta_1^2\left(\frac{d\psi}{d\eta}\right)_1=\left(\frac{2c}{3\kappa(1-\beta)\rho_0}\right)^{\frac{3}{2}}\xi_1^2\left(\frac{dy}{d\xi}\right)_1; \tag{33}$$

$$\alpha\lambda^{\frac{5}{3}}\psi_1^{\frac{5}{2}}=\frac{(\beta\rho_0+y_1-\rho_0)^2-(\beta\rho_0+u_0)^2}{6(1-\beta)\rho_0}. \tag{34}$$

Our object now is to reduce these equations to a simpler set by combining them.

We put $u_0=\gamma\rho_0$ where $\gamma$ is a constant to be determined later. With this value of $u_0$ and making use of (31) and (32), we can reduce (34) to

$$\alpha^2\lambda^{\frac{4}{3}}\psi_1^2=\frac{\beta^2}{9(1-\beta)^2}(2\beta+\gamma)\gamma. \tag{35}$$

Now the quantity $\alpha\lambda^{\frac{2}{3}}\psi_1$ is essentially positive (by (32)). Hence from (35) its value is

$$\alpha\lambda^{\frac{2}{3}}\psi_1=+\frac{\beta}{3(1-\beta)}B(\beta,\gamma), \tag{36}$$

where

$$B(\beta,\gamma)\equiv(2\beta+\gamma)^{\frac{1}{2}}\gamma^{\frac{1}{2}}. \tag{37}$$

Hence by (36), (31), (32) we get the following values of $\lambda$, $y_1$ and $\rho_0$ :—

$$\lambda=\left\{\frac{\beta}{3(1-\beta)}\frac{B}{\alpha}\right\}^{\frac{3}{2}}\frac{1}{\psi_1^{\frac{3}{2}}}; \tag{38}$$

$$y_1=(1+B)\left(\frac{\beta}{3(1-\beta)}\frac{B}{\alpha}\right)^{\frac{3}{2}}; \tag{39}$$

$$\rho_0=\left(\frac{\beta}{3(1-\beta)}\frac{B}{\alpha}\right)^{\frac{3}{2}}. \tag{40}$$

The gas-pressure at the interface is, from (34),

$$p_1=\alpha\lambda^{\frac{5}{3}}\psi_1^{\frac{5}{2}}=\frac{1}{\alpha^{\frac{3}{2}}}\left(\frac{\beta}{3(1-\beta)}B\right)^{\frac{5}{2}}, \tag{41}$$

whilst from (31) the temperature at the interface is

$$T_1=\left\{\frac{3}{a}\left(\frac{B}{3}\right)^{\frac{5}{2}}\left(\frac{\beta}{(1-\beta)\alpha}\right)^{\frac{3}{2}}\right\}^{\frac{1}{4}}. \tag{3.91}$$

The equation (30) can now be reduced to

$$\psi_1^{\frac{1}{3}}\eta_1=\left(\frac{2}{5B}\right)^{\frac{1}{2}}\xi_1, \tag{42}$$

and the equation (33) to

$$\frac{\eta_1^2}{\psi_1^{\frac{1}{3}}}\left(\frac{d\psi}{d\eta}\right)_1=(1+B)\left(\frac{2}{5B}\right)^{\frac{3}{2}}\frac{\xi_1^2}{y_1}\left(\frac{dy}{d\xi}\right)_1. \tag{43}$$

The five equations (30) to (34) have now been reduced to the equivalent set (39) to (43).

The equations (39) to (43) have the following property of invariance. It is known that if $\psi = \phi(\eta)$ is a solution of (29), so also is $\Psi = \mu^4\phi(\mu\eta)$ where $\mu$ is an arbitrary constant. Also if $y = z(\xi)$ is a solution of (24), so also is $Y = \nu \, . \, z(\xi)$, where $\nu$ is an arbitrary constant. Put $\zeta = \mu\eta$. Then on substitution, the equations (42) and (43) become

$$\Psi_1^{\frac{1}{2}}\zeta_1 = \left(\frac{2}{5B}\right)^{\frac{1}{2}}\xi_1 ;$$

$$\frac{\zeta_1^2}{\Psi_1^{\frac{3}{4}}}\left(\frac{d\Psi}{d\zeta}\right)_1 = (1+B)\left(\frac{2}{5B}\right)^{\frac{3}{2}}\frac{\xi_1^2}{Y_1}\left(\frac{dY}{d\xi}\right)_1 ;$$

whatever $\nu$, $\mu$ may be. Hence the equations (42) and (43) are invariant for the two groups of solutions.

## V. *Solutions of Equations of Fit*

We now seek solutions of (42) and (43) when $y(\xi)$ is of the form (25) and $\psi(\eta)$ is, in turn, a centrally condensed, Emden, or collapsed solution of the $n = \frac{3}{2}$ Emden equation. In these two equations, however, the constant $B$ occurs : its value, by (25), is unknown so long as $\rho_0$, the maximum density of matter, is not known. Further, $B$ involves the factor $\frac{1-\beta}{\beta}$, which may have any value from zero to infinity. Hence, theoretically, $B$ also can have any value from zero to infinity. We shall therefore begin by discussing the mathematical problem of the solutions of (42) and (43) when $B$ can have any positive value, and leave the question as to whether or not the resulting configurations can be applied to the stars in nature, to be considered later.

We adopt the graphical method of Cowling.* Let

$$w_1 = -(1+B)\left(\frac{2}{5B}\right)^{\frac{3}{2}}\xi^2\frac{1}{y}\frac{dy}{d\xi} ; \qquad z_1 = \left(\frac{2}{5B}\right)^{\frac{1}{2}}\xi ; \tag{44}$$

$$w^2 = -\frac{\eta^2}{\psi^{\frac{3}{4}}}\left(\frac{d\psi}{d\eta}\right) ; \qquad z_2 = \psi^{\frac{1}{4}}\eta . \tag{45}$$

Then, given a solution $\psi(\eta)$ of Emden's $n = \frac{3}{2}$ equation, we can plot, by (45), a $(w_2, z_2)$ curve corresponding to it. Similarly, by (44), for any given value of $B$, we can plot a $(w_1, z_1)$ curve, making use of Emden's table † for the function

$$y = \frac{\sin\xi}{\xi} .$$

In this way we obtain two sets of curves, and the solution of equations (42) and (43) obviously reduces to determining the points of intersection of a $(w_2, z_2)$ curve with a $(w_1, z_1)$ curve. We exhibit these curves in the diagram.

* T. G. Cowling, *M.N.*, **91**, 472–478, 1931.

† *Handbuch der Astrophysik*, vol. iii, pt. 1, 187, 1930.

The following properties of the $(w_2, z_2)$ curves have been determined by Cowling :—

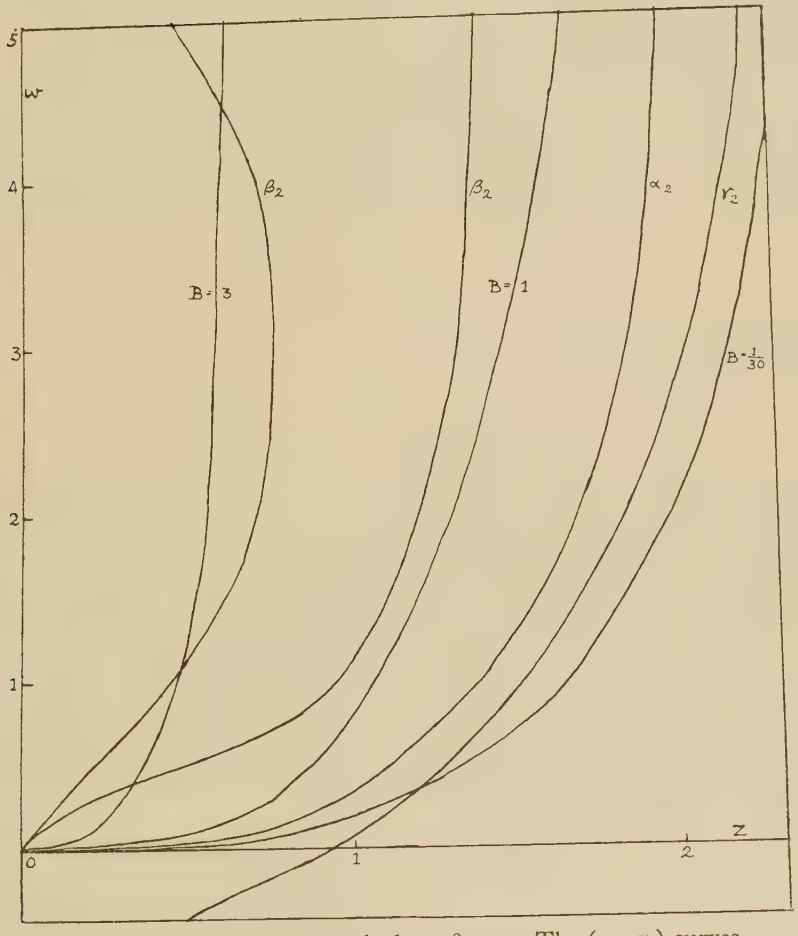
(*a*) All Emden solutions of the $n = \frac{3}{2}$ equation give the same curve (marked $\alpha_2$ in the diagram) which is such that, when $w_2$ and $z_2$ are small, then

$$w_2 = \frac{z_2^3}{3} + o(z^5), \qquad \frac{dw_2}{dz_2} = z_2^2 + o(z^4),$$

and so

$$w_2 = 0, \qquad z_2 = 0, \qquad \frac{dw_2}{dz_2} = 0$$

at the origin.




The $(w_2, z_2)$ curves are marked $\alpha_2$, $\beta_2$, $\gamma_2$. The $(w_1, z_1)$ curves for three values of B are shown.

(*b*) If $\psi(\eta)$ is a centrally condensed solution, the corresponding $(w_2, z_2)$ curve (marked $\beta_2$) lies inside the curve $\alpha_2$ and, at the lower end,

$$w_2 = z_2, \qquad \text{so that } z_2 = 0, \qquad w_2 = 0, \qquad \frac{dw_2}{dz_2} = 1.$$

(*c*) If $\psi(\eta)$ is a collapsed solution, the $(w_2, z_2)$ curve (marked $\gamma_2$) encloses the $\alpha_2$ curve, and, at the lower end, $w_2 \to -\infty$ as $z_2 \to 0$ in such a way that $w_2 z_2{}^3$ is finite.

(*d*) For any $(w_2, z_2)$ curve, at the upper end $w_2 \to +\infty$ as $z_2 \to 0$, so that $w_2 z_2{}^3$ is always finite.

The properties of the $(w_1, z_1)$ curves (marked $B$ in the diagram) which we shall require, are the following :—

(*e*) By Emden's table for $n = 1$, we have that, for any $B$,

$$w_1 \to +\infty \quad \text{as} \quad z_1 \to \left(\frac{2}{5B}\right)^{\frac{1}{2}} (3{\cdot}14159).$$

(*f*) By (25) and (44)

$$w_1 = (1+B)\left(\frac{2}{5B}\right)^{\frac{3}{2}} \frac{\xi}{\sin \xi}(-\xi \cos \xi + \sin \xi).$$

Hence for $z_1$ small, *i.e.* neglecting $z^5$ and higher powers, we have

$$w_1 = \frac{1+B}{12} z_1{}^3, \qquad \frac{dw_1}{dz_1} = \frac{1+B}{4} z_1{}^2,$$

and

$$w_1 = 0, \qquad z_1 = 0, \qquad \frac{dw_1}{dz_1} = 0$$

at the origin.

(*g*) By (*f*) and (44),

$$\frac{dw_1}{dz_1} = (1+B)\left\{z_1{}^2 + \left\{\left(\frac{2}{5B}\right)^{\frac{1}{2}} - z_1 \cot\left(\frac{5B}{2}\right)^{\frac{1}{2}} z_1\right\}^2\right\}.$$

Hence, for positive $B$, $\dfrac{dw_1}{dz_1}$ is always positive, *i.e.* $w_1$ increases monotonically from 0 to $+\infty$ with increasing (positive) $z_1$.

The relative positions and intersections of the $\gamma_2$, $\alpha_2$, $\beta_2$ curves and the $B$ curves can be described as follows. Consider first the intersections of $B$ curves and $\gamma_2$ curves. In virtue of (*c*), (*d*) and (*e*), (*f*), every $B$ curve must cut every $\gamma_2$ curve and can do so once only. This means that, given a collapsed solution of the $n = \frac{3}{2}$ equation in the degenerate-gas zone, the equations of fit (42), (43) can always be solved whatever $B$ may be.

Secondly, consider the intersections of the $B$ curves and the $\alpha_2$ curve. From (*a*) and (*f*) it follows that if

$$\frac{1+B}{4} > 1, \qquad \textit{i.e.}, \; B > 3,$$

then

$$w_2 < w_1 \quad \text{and} \quad \frac{dw_2}{dz_2} < \frac{dw_1}{dz_1}$$

for equal $z_2$, $z_1$.

Hence if $B > 3$, the $B$ curve lies inside the $\alpha_2$ curve initially, and therefore intersects it once. If $B < 3$ the $B$ curve lies outside the $\alpha_2$ curve initially and therefore, if it cuts it at all, it must do so twice. But for sufficiently small values of $B$, the $B$ curve may not intersect the $\alpha_2$ curve at all.

Lastly we have the intersections of $\beta_2$ and $B$ curves. By (*b*) and (*f*) it

follows that every $B$ curve lies initially below a $\beta_2$ curve. Hence, if a $B$ curve cuts a $\beta_2$ curve it must do so twice. Also, it is always possible, given a $\beta_2$ curve, to find a value of $B$ so large that the corresponding $B$ curve cuts the given $\beta_2$ curve. It is only, in fact, for the values of $B$ lying between 0 and 3 that the intersections of $B$ curves and $\beta_2$ curves may be impossible, since it is for some of these values that the $B$ curves may lie altogether outside the $\alpha_2$ curve.

We can say, therefore, that given an Emden solution of the $n=\frac{3}{2}$ equation, the equations (42), (43) possess one solution if $B>3$, but if $B<3$ they have either two solutions or none at all. On the other hand, given a centrally condensed solution of the $n=\frac{3}{2}$ equation, there will be a certain number, $\omega_0$, such that for $B \geqslant \omega_0$ (42) and (43) will have two solutions, whilst for $B<\omega_0$ they will have none. Also, $\omega_0$ will depend on the particular centrally condensed solution which is given, and the lower limit of all the numbers $\omega_0$ will lie between 3 and 0.

Once the equations of fit (42), (43) have been solved for any value of $B$, the radiation-density at the centre can be obtained from (39). For if $u_c$ is the value of $u$ at the centre, we have by (22) and (25) that

$$u_c+\rho_0=A,$$

and hence by (39),

$$u_c+\rho_0=\frac{\xi_1}{\sin\xi_1}(1+B)\left(\frac{\beta}{3(1-\beta)}\frac{B}{\alpha}\right)^{\frac{3}{2}}, \tag{45.1}$$

or, using (40),

$$u_c=\left(\frac{\beta}{3(1-\beta)}\frac{B}{\alpha}\right)^{\frac{3}{2}}\left(\frac{(1+B)\xi_1}{\sin\xi_1}-1\right).$$

From this it follows that the radiation-density is always finite at the centre, since by (31) the function $\sin\xi_1/\xi_1$ can never be zero unless $\rho_0$ is zero, which is not admissible.

We can now calculate the ratio of the mass due to matter to that due to radiation in the core inside radius $r$. We have

$$M_0(r)=\int_0^r r^2\rho_0 dr \quad \text{and} \quad M_1(r)=\int_0^r r^2 u dr.$$

Using (22) and (25) we find

$$\frac{M_1(r)}{M_0(r)}=\frac{A(\sin\xi-\xi\cos\xi)-\frac{1}{3}\xi^3\rho_0}{\frac{1}{3}\xi^3\rho_0}.$$

Hence, inserting the value of $A$,

$$\frac{M_1(r)}{M_0(r)}=\frac{3(u_c+\rho_0)}{\rho_0}\left(\frac{\sin\xi}{\xi^3}-\frac{\cos\xi}{\xi^2}\right)-1, \tag{45.2}$$

where $\xi$, $r$ are related by (23).

For small values of $\xi$ (*i.e.* neglecting $\xi^2$, etc.) we get from (45.2)

$$\frac{M_1(r)}{M_0(r)}=\frac{1}{4}\frac{u_c+\rho_0}{\rho_0}-1. \tag{45.3}$$

For our composite models, using (45.1), this becomes

$$\frac{M_1(r)}{M_0(r)} = \frac{1+B}{4}\frac{\xi_1}{\sin\xi_1} - 1. \tag{45.4}$$

We therefore have the result : if the core of a composite configuration is small, so that $\xi_1/\sin\xi_1$ is approximately unity (*i.e.* when $\xi_1{}^2$, etc., can be neglected), then the least value of $B$ admissible is 3, since, by (45.4), for lesser values of $B$ we should get negative values of $M_1/M_0$, which cannot be allowed.

This point is of interest in connection with actual giant stars (§ VII).

## VI. *Composite Configurations*

We shall now attempt to answer the question : Given a solution (of Emden's $n=3$ equation) for the outer perfect-gas envelope, what values of $B$ are possible, if the configuration is to consist of the perfect-gas envelope, a degenerate-gas zone and a core of the type we are considering ?

Milne * has shown that in the perfect-gas envelope the solution is centrally condensed, Emden, or collapsed according as

$$\frac{1}{C^{\frac{1}{2}}} \lessgtr 2{\cdot}0150,$$

where

$$C = \frac{64}{\frac{1}{3}a}\left(\frac{R}{\mu c^2}\right)^4\left(\frac{2c}{\kappa}\right)^3\frac{1-\beta}{\beta^4 M^2}. \tag{46}$$

$M$ is the total mass (in ergs) of the configuration ; $R$ is the perfect-gas constant ; $\mu$ is the molecular weight ; and $a$ is Stefan's constant. Further, Milne † and Cowling ‡ have shown that the type of the solution in the envelope implies the type in the degenerate-gas zone according to the following scheme :—

| *Perfect-gas envelope* ($n=3$) | *Degenerate-gas zone* ($n=\frac{3}{2}$) |
|---|---|
| (1) Collapsed. | Collapsed or Emden. |
| (2) Emden. | No degenerate-gas zone. |
| (3) Centrally condensed. | Centrally condensed. |

In case (1) the value of $C$ does not determine the type of solution uniquely in the degenerate zone : there are two types possible in this zone, according as the density of matter near the centre is greater or less than the maximum density of matter, $\rho_0$.

In case (3), which interests us more particularly, the value of $C$ determines the *type* of solution in the degenerate zone, but not the *particular* centrally condensed solution of the $n=\frac{3}{2}$ Emden equation. There are a number of such solutions possible for given $C$, and each one will give a

* E. A. Milne, *loc. cit.*, § 14.
† E. A. Milne, *loc. cit.*, § 28.
‡ T. G. Cowling, *loc. cit.*

different $\beta_2$ curve on our diagram. However, it follows from inspection of Cowling's * diagram that the $\beta_2$ curve of this group lying nearest to the $\alpha_2$ curve is uniquely determined. We call it the "maximum $\beta_2$ curve" of the group, and its position on our diagram is therefore uniquely fixed by the given value of $C$.

The maximum density of matter, $\rho_0$, is a definite number to be determined by physical theory. For our present purpose, we can take it to be a known constant. Then the value of $B$ by (40) is

$$B = 3\alpha\rho_0^{\frac{2}{3}}\frac{(1-\beta)}{\beta}. \qquad (47)$$

The different models are distinguished by different values of $\beta$ and $M$. We see therefore from (47) and (46), that whereas the value of $B$ varies with $(1-\beta)/\beta$ as we go from model to model, the value of $C$ varies with $(1-\beta)/\beta^4M^2$.

Now consider the various possibilities.

(1.1) *Collapsed solutions in both envelope and degenerate-gas zone.*

We have seen that every $B$ curve cuts every $\gamma_2$ curve. Hence for any pair of values of $\beta$, $M$ satisfying $1/C^{\frac{1}{2}}>2{\cdot}0150$, the model can have a core of incompressible matter and radiation.

(1.2) *Collapsed solution in envelope and Emden solution in degenerate-gas zone ;* or

(3) *Solutions in both regions centrally condensed.*

We have seen that not every $B$ curve cuts the $\alpha_2$ curve or every $\beta_2$ curve. Hence for every $1/C^{\frac{1}{2}} \leqslant 2{\cdot}0150$ there will be pairs of values of $\beta$, $M$, for which the $B$ curve will not intersect the corresponding $\alpha_2$ or maximum $\beta_2$ curve. But, of course, given $C$ alone, $\beta$ and $M$ can always be adjusted so that the $B$ curve and the $\alpha_2$ or maximum $\beta_2$ curve will intersect.

This result is only of importance in case (3), since in case (1.2), if we failed to find a suitable value of $B$ for given $\beta$ and $M$, we could always continue the Emden solution in the degenerate-gas zone to the centre and suppose that there was no core. This is not possible in case (3). Hence for certain values of $\beta$, $M$ a centrally condensed model cannot have a core of the type we are considering. It therefore becomes important to try to determine these excluded values of $\beta$, $M$ for given $C$.

Unfortunately, no exact solution of this problem can be given at the present time. Tables of the centrally condensed solutions of both Emden's $n=3$ and $n=\frac{3}{2}$ equations would be required in order to determine the maximum $\beta_2$ curve for given $C$. Such tables do not yet exist. Supposing this could be done, however, then a value of the factor $(1-\beta)/\beta$ in (47) could be found by drawing $B$ curves on our diagram, such that for lesser values of $(1-\beta)/\beta$ the $B$ curves would no longer intersect the given maximum $\beta_2$ curve. In a general way it can be said that if the factor $(1-\beta)/\beta$ for the model is small compared with unity (so that $B<3$) no intersection will be possible, but if it is large compared with unity we shall get intersections.

* T. G. Cowling, *loc. cit.* Fig. 2.

## VII. *Application to Giant Stars*

We seek to find out if case (3) can be applied to actual giant stars. This is equivalent to discovering whether or not the values of $\beta$, $M$ actually found amongst these stars form excluded pairs of values for the calculated value of $C$ for the star. We know observationally that the values of $M$ do not differ much from the value for the Sun. As to $\beta$, since $k$ is unknown, we can only say that its value must be small, since the luminosities of giant stars are great.

Suppose we consider a star of the Sun's mass, which is only slightly centrally condensed, so that the value of $1/C^1$ for the star is, say, equal to 2. We can then say that its maximum $\beta_2$ curve lies close to the $\alpha_2$ curve in our diagram: for definiteness suppose the greatest value of $z_1$ for the curve is $>1$. We now calculate $\beta$ from (46) and $B$ from (47), and show that the resulting $B$ curve lies well inside the $\alpha_2$ curve. We have

$$C=\tfrac{1}{4}; \qquad \frac{R}{\mu c^2}=4{\cdot}6\times 10^{-14}; \qquad \frac{\kappa}{2c}=10^{-48};$$

$$M=1{\cdot}8\times 10^{54}\text{ ergs}; \qquad \tfrac{1}{3}a=2{\cdot}55\times 10^{-15}.$$

We thus obtain, from (46), $\beta=10^{-4}$ approximately, and hence

$$(1-\beta)/\beta=10^{+4} \text{ approximately.}$$

Again, Milne * has shown that, at the interface between perfect and degenerate gas, the gas-density is

$$\rho'=3{\cdot}66\times 10^{7}(1-\beta)/\beta \text{ grams/cm.}^3.$$

Hence in our model

$$\rho'=3{\cdot}66\times 10^{11} \text{ grams/cm.}^3.$$

From this we can say that, whatever the maximum density of matter is, it must be greater than $3{\cdot}66\times 10^{11}$ grams/cm.$^3$. Let us take the most unfavourable case, from our point of view, and assume that the maximum density of matter is only a little greater than this, say, $10^{12}$ grams/cm.$^3$ Then

$$\rho_0=10^{12}\,.\,c^2=10^{33} \text{ ergs/cm.}^3.$$

Hence, by (47), substituting for $\alpha$ and $(1-\beta)/\beta$, we have

$$B=3\times(3{\cdot}8)\times 10^{-23}\times 10^{22}\times 10^{4}=10^{4} \quad \text{approx.} \quad . \qquad . \qquad (48)$$

By (*e*), § V, we know that, for any value of $B$, the corresponding $B$ curve lies between the lines $z_1=0$ and $z_1=(2/5B)^{\frac{1}{2}}(3{\cdot}14159)$. Hence the greatest value of $z_1$ for the $B$ curve under consideration is, by (48),

$$z_1=2^{\frac{1}{2}}\times 3{\cdot}14159/5^{\frac{1}{2}}\times 10^{2}=2\times 10^{-2} \quad \text{approx.}$$

On our diagram this $B$ curve is almost indistinguishable from the line $z_1=0$. It must therefore cut the $\beta_2$ curve for the problem.

Hence for a star of the Sun's mass and of very high luminosity a core of

* E. A. Milne, *loc. cit.*, § 26.

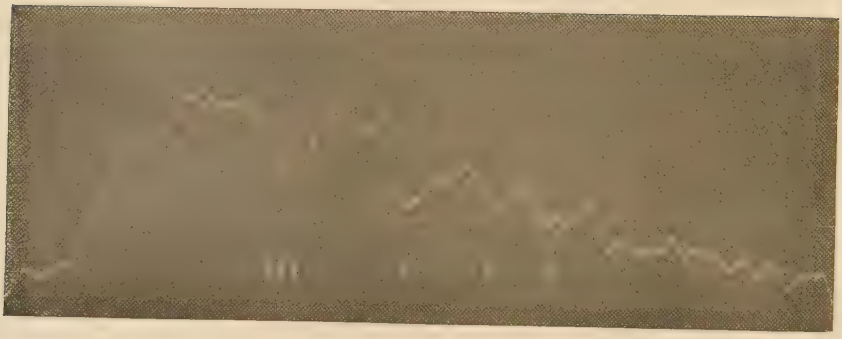


Fig. 1.—Microphotometric Tracing of van Maanen No. 1166.

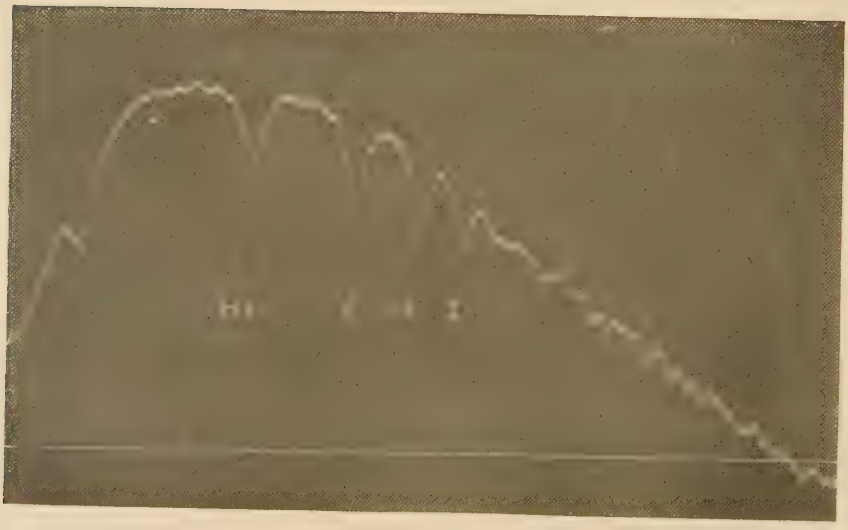


Fig. 2.—Microphotometric Tracing of Companion $o_2$ Eridani.

incompressible matter and radiation can exist within the degenerate zone, provided the degree of " central condensation " is small. The central values of the gas-density and radiation-density are then both finite.

Further, the value of $\xi_1$ obtained as the solution of (42), (43) must be very small compared with unity. Hence (45.3) for the ratio $M_1/M_0$ holds throughout the core, and by (45.4) the value of this ratio is of the order of $10^4$. The mass of the radiation in the core is therefore much larger than that of the matter.

*The University,*
*Leeds.*

---

## *Note on the Spectrum of a Probable White Dwarf in h Persei.*

By Yngve Öhman, Ph.D. (Plate 1.)

(*Communicated by the Secretaries.*)

P. Th. Oosterhoff * has recently drawn attention to a star in the double cluster *h* and $\chi$ Persei which seems to be a white dwarf. The star, which is identical with van Maanen's No. 1166,† has a very great proper motion ($\mu = 0''{\cdot}166$), though it is rather faint ($m_{\text{phot}} = 13^{\text{m}}{\cdot}44$), and has a negative colour index ($-0^{\text{m}}{\cdot}10$). If the star were an ordinary A-type star the linear velocity would be about 2000 km./sec., and as this seems very improbable Oosterhoff supposes the star to be a white dwarf.

In connection with spectrophotometric work on stars in *h* and $\chi$ Persei, which I have started at the new Stockholm Observatory, I have taken two spectra of this interesting star. A plate taken on 12th October with the 40-cm. Zeiss astrograph combined with objective prism shows a faint spectral image of the star. The dispersion is short, only 1·05 mm. between $H\gamma$ and $H\delta$. The exposure time was $2^{\text{h}}\ 30^{\text{m}}$. Though the spectral image is overlapped by other images at the ends, its central part is well defined and has been analysed by means of a Koch-Goos microphotometer. The tracing shows the $H\gamma$ and $H\delta$ lines very wide and diffuse. It is thus obvious that the star must be an A-type star.

On 5th November I succeeded in getting a spectrum with the 100-cm. Grubb reflecting telescope in conjunction with a slitless quartz spectrograph from Zeiss. The spectrum taken with this instrument is still shorter, only 0·82 mm. between $H\gamma$ and $H\delta$. The exposure time was also in this case $2^{\text{h}}\ 30^{\text{m}}$. The image is faint, but a microphotometric tracing shows the hydrogen lines very well, as is evident from fig. 1. The lines are extremely wide, but this is to some extent due to the fact that a rather wide diaphragm has been used in the photometer. The depression in the curve between $H\gamma$ and $H\delta$ is not real, as it is not present on the astrographic plate. The hydrogen lines can be followed to $H\zeta$, but not farther. In the ultra-violet

* *Bull. Astr. Inst. Netherlands*, vi, No. 206, 1930.
† *Mount Wilson Contribution*, No. 205, 1920.

the image is, of course, too faint to allow any definite analysis, but there does not seem to be any continuous hydrogen absorption. This agrees well with the negative colour index. The K line is not present in the tracing and the star seems thus to be of the spectral type A0.

A comparison with the spectrum of Companion $o_2$ Eridani (fig. 2) is of great interest. Professor Lindblad has kindly put at my disposal a spectrum of this star which was taken by him in 1921 with the two-prism quartz spectrograph of the Lick Observatory. As has already been pointed out by Lindblad,* and is very apparent from fig. 2, the hydrogen lines are extremely wide, but only the earlier members of the Balmer series are present. $H\eta$ is hardly visible and higher members cannot be seen. (This is surely a consequence of the high pressure in the stellar atmosphere.) The spectrum is also remarkably luminous in the ultra-violet and does not show any continuous hydrogen absorption.

As far as we can see from a comparison between the two tracings, van Maanen's No. 1166 has a spectrum very similar to that of Companion $o_2$ Eridani. We have thus a strong confirmation of the suggestion made by Oosterhoff that the star really is a white dwarf. A determination of the parallax will settle the question definitively.

*Stockholm Observatory :*
1931 *November* 9

---

*A Comparison-Image Micrometer.* By F. J. Hargreaves.

The following is an account of an instrument for measuring the separations and position angles of double stars, which appears to possess certain advantages over the filar micrometer, particularly in that it is less subject to systematic error and personal idiosyncrasy, and gives more consistent results.

Briefly, the instrument is such that the images of two artificial stars are projected into the field of view of the telescope, side by side with the images of the double star to be measured, and the distance between the artificial stars is varied until the separation of their images appears the same as that of the images of the real stars. The artificial stars are mounted on a rotatable carrier so that the line joining their images can be placed parallel to the line joining the real star images, thus enabling position angles to be measured, and there is provision for varying the brightness of both artificial stars independently so that their images match those of the real stars.

The image of the artificial double star is formed by a lens A (fig. 1), of short focal length (about 2 inches), and is reflected into the focal plane of the telescope by a reflecting-transmitting cube B consisting of two 45°–90° prisms cemented together, one of which has its hypotenuse surface lightly silvered. There is serious loss of light by absorption at the silvered surface, but a method of avoiding this has been devised.

The artificial star C (fig. 2) is mounted on the telescope tube and consists

* *Mount Wilson Contribution,* No. 228, 1921.

*Fig. 1.*

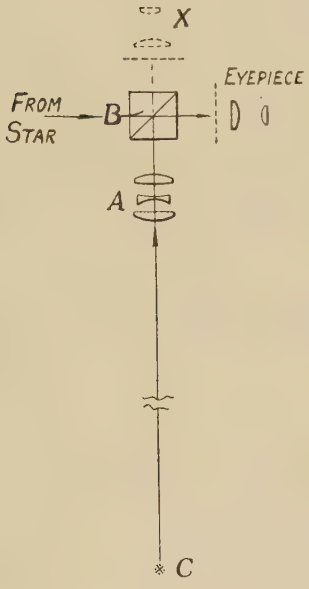



*Fig. 2.*

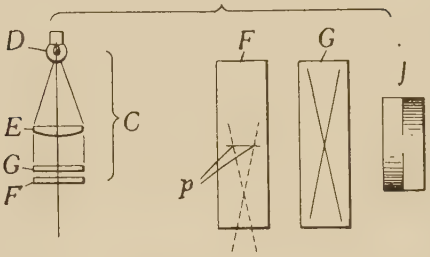



*Fig. 3.*

*Fig. 4.*

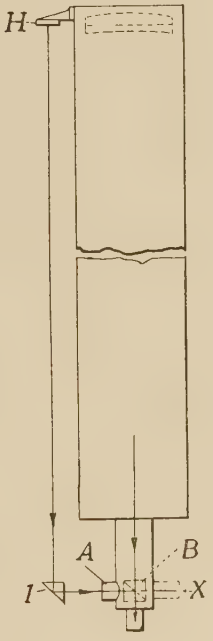



*Fig. 5.*

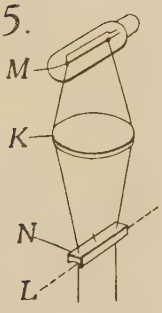
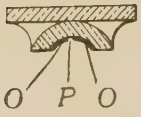



*Fig. 6.*



of a lamp D and condenser E and two slit plates F and G. The plate F has a single slit and the other has two slits crossing one another at an acute angle. The plate F is stationary and the plate G slides across it in a straight line perpendicular to the slit in F, and it will be clear that the light can pass through the slits only at the two points *p* at which they cross, and that the distance between these points will be proportional to the distance between the fixed slit and the point of intersection of the two movable slits. The position of this point of intersection cannot be directly determined with accuracy, and therefore two settings are made at opposite sides of the point of intersection, and the actual reading is the distance travelled by the plate G in passing from one setting to the other. This is analogous to the "double-distance" method of using the filar micrometer.

The dimensions and proportions of the arrangement are such that one centimetre of "double distance" corresponds approximately to one second of arc. The scale is therefore a very open one and can be read (by estimation of tenths of millimetres) to an adequate degree of accuracy without a vernier, divided screw head or other auxiliary device.

The slits are made by cutting through a blackened photographic emulsion coating by means of a razor guided by a straight-edge. In cutting the two intersecting slits care was taken to use the same part of the straight-edge, in order to minimise the effect of any departure from straightness. Accuracy depends on the strict proportionality of the distance between the crossing points *p* and the distance travelled by the plate G, but it is highly probable that the precautions taken in cutting the slits have ensured that any error in this proportionality is much less than the accidental errors in the observations. The method of determining the constant of the instrument was devised to reveal any error of this kind, but none was found.

It will be seen that the instrument bears a superficial resemblance to certain range-finders and gun-sights in which the image of a graticule is formed in the focus of a telescope; but in these instruments the variation in size of the image is brought about by varying the focal length of the projecting system. This limits the range of variation severely and introduces sources of error which are avoided by the arrangement here described.

Fig. 3 shows how the components are arranged on the reflector tube. The small mirror H is for the purpose of increasing the effective distance between the slits and the small lens A (to about 10 feet), and also to enable the slits to be mounted in a position convenient for manipulation. The whole assembly of slits, lamp and condenser is mounted to rotate about the optical axis, and a circle for reading position angles is provided. For a refractor it would be necessary to provide a second mirror or prism I (fig. 4) to direct the light laterally into the lens A.

It has been found essential to adjust the magnitudes of the artificial stars to match those of the real stars, and for this purpose a double grey wedge *j* (fig. 2) is mounted to slide across the slit F so that the line of demarcation between the two wedges accurately intersects the slits G at their point of intersection. Thus only one of the two artificial stars is dimmed. Another wedge covering both artificial stars is provided, and the images of bright

real stars are dimmed by black gauze screens over the mouth of the telescope tube.

If the magnitudes do not match there is a marked systematic error in the measurement of separation; when the artificial stars are brighter than the real stars the readings tend to be too low, and *vice versa*.

Colour differences do not seem to introduce any systematic error, but a series of colour filters has been made so that a rough colour match can be obtained. The use of these filters eliminates a source of error, theoretically present, due to the secondary spectrum produced by the lens A. This lens, an anastigmatic triplet, is very highly corrected, and no trace of chromatic aberration can be seen in the images.

The lens is stopped down (from F/3·5) to F/6, which is the same relative aperture as that of the telescope mirror. The object of this is to ensure that the diffraction patterns in the two pairs of images shall be exactly the same size. The width of the slits is such that the diameter of the geometrical images of the artificial stars is a small fraction of the diameter of the spurious discs.

### *Determination of Constant*

The upper limit of separation of this instrument is about 12″. This makes it impossible to use the customary transit method of determining the constant. It was thought advisable to use a fundamental method rather than to place reliance on previous determinations of the separation of pairs less than 12″. Accordingly, a "standard star" of known linear separation was set up at a distance of about 1600 feet (determined accurately with a theodolite) and "measured" exactly as are actual stars.

Two independent determinations were made on different nights, with two slightly different standards. One of these consisted of five ¼-inch steel balls placed in contact in a row and illuminated from a distance of a few yards by means of a motor-car head-lamp. A shutter was provided so that an assistant could expose any two of the balls, and by manipulating this shutter, double stars of four different separations were obtained, namely, 1 inch, 0·75 inch, 0·5 inch, and 0·25 inch. The readings were as follows:—

| cm. | cm. | cm. | cm. |
|---|---|---|---|
| 1 inch: 9·43 | 0·75 inch: 6·99 | 0·5 inch: 4·73 | 0·25 inch: 2·42 |
| 9·35 | 7·05 | 4·74 | 2·44 |
| 9·42 | 7·05 | 4·77 | 2·45 |
| Mean 9·40 | Mean 7·03 | Mean 4·747 | Mean 2·437 |

The settings for the 0·25-inch separation were interfered with on one side by a particle of dust on one of the slits which could not be dislodged at the time; the readings on the other side give a mean of 2″·36. The weighted mean of the various settings gave a constant of 1″·155 per cm.

Later, a standard consisting of two 1-inch balls was used. This was made rotatable about the line of sight to discover whether there was any systematic difference due to different orientations, and whether any difference was

caused by the position of the comparison images above or below the standard-star images. The results were as follows:—

| (1) Horizontal. Comparison Images Above. | (2) Horizontal. Comparison Images Below. | (3) Vertical. Comparison Images on Left. |
|---|---|---|
| cm. | cm. | cm. |
| 9·45 | 9·38 | 9·40 |
| 9·44 | 9·39 | 9·40 |
| 9·45 | 9·41 | 9·40 |
| Mean 9·447 | Mean 9·393 | Mean 9·400 |

The unweighted mean of these settings gave a constant of 1″·155, confirming the previous determination. There is a difference between the means of the first and second series of settings, but this is of opposite sign to a suspected systematic difference "above–below" in the settings on $\gamma$ Leonis made on certain nights. It has since been concluded that there is no appreciable systematic difference due to this cause, but any such source of error can easily be eliminated by making equal numbers of settings in both positions.

It is to be noted that the determination of the constant involved observations of precisely the same nature as the observations of actual stars. Consequently, if there is any systematic error affecting all settings, due to the nature of the instrument (this does not seem likely), such error will not appear in the results.

*Separation*

Below is a list of measurements of separation of fourteen stars made with this instrument fitted to a 12¼-inch reflector, together with measurements of ten of the stars made with the filar micrometer during the same period (1931 May to October), by the Rev. T. E. R. Phillips, which can be taken as the best that can be done with that instrument. The quantities in columns (*a*) and (*b*) are the average deviations from the means, from *night* to *night* and from *setting* to *setting* respectively. "C.-I." signifies Comparison-Image micrometer, and "F" Filar micrometer.

As will be seen, the scatter from night to night, and from setting to setting, is considerably less with the new instrument than with the filar micrometer. In the case of the night-to-night scatter, the ratio is in the mean about 7 : 1, and from setting to setting about 3 : 1. It is the writer's belief that if the instruments had been interchanged between the two observers these ratios would have been of the order of 70 : 1 and 30 : 1 respectively.

Measurements were made on many nights when the definition would ordinarily be considered quite unsuitable for micrometrical work, in order to test the instrument as severely as possible. It is generally admitted that the reflector is less suitable for double-star measurement than the refractor, and it is much to be desired that a refractor in the hands of an experienced observer should be provided with an instrument such as is here described.

| Star | | Mean Separation | | | | Mean Deviations | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | *a* | | *b* | |
| | | C.-I. | No. of Nights | F. | No. of Nights | C.-I. | F. | C.-I. | F. |
| | | ″ | | ″ | | ″ | ″ | ″ | ″ |
| Σ2727 | γ Del | 10·36 | 6 | ... | ... | ±0·022 | ... | ±0·030 | ... |
| Σ2978 | | 8·22 | 4 | ... | ... | ·030 | ... | ·033 | ... |
| Σ2264 | 95 Her | 6·40 | 7 | 6·31 | 6 | ·008 | ±0·058 | ·032 | ±0·080 |
| Σ1670 | γ Vir | 5·79 | 8 | 5·80 | 6 | ·025 | ·104 | ·040 | ·083 |
| Σ2140 | α Her | 4·65 | 6 | 4·62 | 5 | ·016 | ·076 | ·032 | ·081 |
| Σ1424 | γ Leo | 3·98 | 9 | 3·87 | 3 | ·013 | ·028 | ·033 | ·041 |
| | | 3·97 | 8 | ... | ... | ·008 | ... | ·033 | ... |
| Σ2382 | ε¹ Lyr | 2·87 | 12 | 2·94 | 4 | ·024 | ·055 | ·026 | ·078 |
| Σ1977 | ε Boo | 2·77 | 8 | 2·80 | 5 | ·008 | ·036 | ·019 | ·078 |
| Σ2383 | ε² Lyr | 2·28 | 10 | 2·27 | 4 | ·003 | ·010 | ·019 | ·064 |
| Σ2054 | | 1·15 | 3 | ... | ... | ·004 | ... | ·007 | ... |
| Σ1865 | ζ Boo | 0·98 | 5 | 1·05 | 6 | ·005 | ·056 | ·010 | ·066 |
| OΣ395 | 16 Vul | 0·80 | 3 | ... | ... | ·006 | ... | ·010 | ... |
| Σ1937 | η CrB | 0·62 | 5 | 0·56 | 10 | ·005 | ·048 | ·008 | ·068 |
| Σ73 | 36 And | 0·60 | 3 | 0·54 | 7 | ·001 | ·060 | ·011 | ·049 |

Two nights' observations only have been rejected, namely, the first observations of $\epsilon^1$ and $\epsilon^2$ Lyræ made under bad conditions when the stars were at a low altitude. Both were well outside the range of scatter of all subsequent observations, and although their inclusion would have altered the means slightly in the direction of closer agreement with the filar micrometer values, it was thought best to omit them.

The comparatively large scatter in the case of $\epsilon^1$ Lyræ is unaccountable, and compares strangely with the more accordant results with $\epsilon^2$ Lyræ, only 3′ distant. The scatter is well distributed over the series, both in separation and position angle.

Two figures are given for γ Leonis. The first includes a very discrepant series of observations made on June 3 by the writer and Dr. W. H. Steavenson in fairly good seeing. Although the two observers confirmed one another's figures closely, the mean for this night was no less than 0″·1 higher than the mean for all other nights, and this single large discrepancy cannot be accounted for.

### *Position Angle*

There are several methods of measuring position angle; one may place the artificial images alternately on opposite sides of the real images, or one may place them all in line, or the plate G may be removed altogether and the slit F used alone in the same way as a position web. The first method gives perhaps the most consistent results, but the final results of all three methods are in very good agreement. There is usually a large difference between the readings taken on opposite sides of the real images in the case of unequal

pairs, as is to be expected. Sometimes, though not always, this difference diminishes as successive settings are made. It was expected that this difference would be reduced by placing the bright component of the artificial star opposite the faint component of the real pair, but in general this was not so.

The list below, with filar determinations for comparison, demonstrates that, on balance, there is some advantage in favour of the new instrument, but this advantage is not nearly so great as in the case of separation. The small scatter in the case of Σ73 (separation 0″·60) is probably the result of chance and cannot be regarded as representative of what is likely to occur generally with such close pairs.*

| Star | | Mean Position Angle | | | | Mean Deviations *a* | | *b* | |
|---|---|---|---|---|---|---|---|---|---|
| | | C.-I. | No. of Nights | F. | No. of Nights | C.-I. | F. | C.-I. | F. |
| | | ° | | ° | | ° | ° | ° | ° |
| Σ2727 | γ Del | 269·9 | 6 | ... | ... | ±0·27 | ... | ±0·19 | ... |
| Σ2978 | | 146·2 | 4 | ... | ... | 0·09 | ... | 0·25 | ... |
| Σ2264 | 95 Her | 259·4 | 7 | 259·0 | 6 | 0·13 | ±0·38 | 0·11 | ±0·30 |
| Σ1670 | γ Vir | 318·5 | 3 | 318·9 | 6 | 0·13 | 0·33 | 0·17 | 0·36 |
| Σ2140 | α Her | 110·8 | 8 | 109·6 | 5 | 0·70 | 0·36 | 0·19 | 0·26 |
| Σ1424 | γ Leo | 119·4 | 1 | 118·2 | 3 | ... | 0·23 | 0·19 | 0·32 |
| Σ2382 | $\varepsilon^1$ Lyr | 6·8 | 16 | 5·6 | 4 | 0·60 | 0·18 | 0·19 | 0·45 |
| Σ1877 | ε Boo | 334·1 | 3 | 333·6 | 5 | 0·07 | 0·16 | 0·30 | 0·49 |
| Σ2383 | $\varepsilon^2$ Lyr | 113·0 | 13 | 114·0 | 4 | 0·33 | 0·23 | 0·19 | 0·20 |
| Σ2054 | | 355·4 | 3 | ... | ... | 0·24 | ... | 0·54 | ... |
| Σ1865 | ζ Boo | 134·5 | 1 | 133·7 | 6 | ... | 1·03 | 0·33 | 0·55 |
| OΣ395 | 16 Vul | 108·8 | 3 | ... | ... | 0·49 | ... | 0·37 | ... |
| Σ1937 | η CrB | 208·2 | 4 | 204·0 | 10 | 0·42 | 1·53 | 0·65 | 0·68 |
| Σ73 | 36 And | 88·9 | 3 | 85·4 | 7 | 0·17 | 1·50 | 0·26 | 0·89 |

The zero of position angle is determined by first setting a web to the path of a star in transit, and then setting the image of the slit F into coincidence with the web. Exact coincidence is revealed by a distinctive diffraction effect, which is so sensitive that little accuracy is lost by the addition of this second stage to the usual procedure.

### *General*

The comparison images are, of course, quite steady, while the real images are oscillating. If the oscillation is continuous it is impossible generally to make a direct comparison of separation, but instead one judges whether the four images are at the corners of a parallelogram. The oscillation causes this parallelogram to alter its shape continually, but this does not prevent the

* Since this was written, observations have been obtained on six further nights. The mean of all the observations is 88°·6, and the mean derivations are (*a*) ±0°·47, (*b*) ±0°·42.

detection of lack of parallelism. When the components are very close and the oscillation has an amplitude greater than the separation, this " parallelogram " method is impracticable, but in such cases it is easy to make a direct comparison owing to the closeness of the images.

In bad seeing, such as causes the real images to expand into hazy discs, there is a tendency to measure the " overall dimensions " of the pair rather than the distance between centres, resulting in a setting of the comparison images too far apart. Once this tendency is recognised it is easy to guard against it.

The difference in the appearances of the two pairs of images on all but the best nights, due to atmospheric disturbance, is undoubtedly troublesome, but after a little practice the difficulty diminishes. In this respect the instrument is not so easy to use as those in which the actual images are duplicated, such as the heliometer and the tilting grating advocated by Lawrence Richardson.* In the heliometer, however, the linear distance to be measured is the same as the actual linear separation of the two images, whereas in the present instrument it is about one thousand times as great. The measurements of separation by the grating are subject to uncertainty owing to the differing colours of stars, and this uncertainty can be removed (by narrow transmission filters) only at the cost of very great loss of light. Moreover, the accuracy of the setting depends on the estimation of bisection, which is subject to large personal differences, while inequality of magnitude is a cause of uncertainty. Both the heliometer and the tilting grating are unsuitable for measuring very small separations close to the resolving limit of the telescope, and it is in such cases that the present instrument shows to best advantage.

### *Future Modifications*

The serious loss of light by absorption at the thinly silvered surface restricts the scope of the present instrument in the direction of faint stars. This could be avoided by placing the eyepiece at X (fig. 1) and increasing the thickness of the silver coating. The brightness of the comparison images would then be diminished, but this would not matter for faint stars. Neither arrangement, however, is suitable for pairs which differ very greatly in magnitude, because the brighter component may have to be cut down, to match a not very bright artificial image, to such an extent that the fainter component is lost. It seems, therefore, that not only must provision be made for placing the eyepiece in either position according to circumstances, but also the brightness of the artificial stars must be increased far beyond what can be obtained by a condenser or collimator with the source of light at its principal focus. An arrangement for this purpose is shown in fig. 5. A lens K forms an image at L of a straight filament M. A very short-focus, divergent, cylindrical lens N is placed just within the focus L so that the light is collimated in the form of a very much flattened parallel beam. A slit placed in this beam will evidently pass a much greater proportion of the light incident on the lens K than if it were placed in a cylindrical beam having the cross-section of the

* *B.A.A. Journ.*, **35**, 155, 1925; *Ast. Nach.*, **139**, No. 3335, p. 353, 1896.

aperture of the lens, as in the existing arrangement. Loss of light owing to aberrations in the cylindrical lens would be minimised by placing the slit actually on the lower surface of the cylindrical lens. This lens would not be easy to make by ordinary methods, but a satisfactory one has been improvised by cementing a " slice " of glass tubing to a thin flat glass plate, as shown (enlarged) in fig. 6, which also shows the slit P formed by cutting through a blackened gelatine coating O (or a silver film) on the concave surface. An arrangement of this kind has been rigged up for experiment and shows considerable promise, but so far it has not been carried into effect owing to the fact that it has been found impossible to obtain a small incandescent lamp with a straight filament.

A modified form of the instrument is now under construction for measuring the diameters of planets, and will be the subject of a later communication.

Acknowledgments are gratefully made to Mr. B. M. Peek, Capt. M. A. Ainslie and the Rev. T. E. R. Phillips ; the first for suggesting the use of artificial stars instead of a system of bright lines originally proposed, and for stressing the importance of making the aperture ratio of the projecting lens the same as that of the telescope ; the second for the device of slits for obtaining a variable-separation artificial double star ; and the third for providing the material for the comparison with the filar micrometer.

---

## *On a Criterion for the Prediction of an Unknown Planet.*

By Ernest W. Brown, Sc.D., F.R.S.

1. *Summary.*—The use of the formula

$$\sin(x+\Delta x)+\sin(x-\Delta x)-2\sin x=-(4\sin^2\tfrac{1}{2}\Delta x)\sin x, \qquad (1.1)$$

or in its more compact form,

$$\Delta^2 \sin x+(4\sin^2\tfrac{1}{2}\Delta x)\sin x=0,$$

for the determination of unknown periodicities from a continuous set of observations has been proposed on several occasions. In this paper I am going to show, first, how it can be *effectively* used to eliminate from such a set a periodic term with known period but unknown amplitude and phase, and then, that the same transformation, applied to the perturbations in longitude of a planet produced by the attraction of an exterior planet, gives to the curve obtained by plotting these perturbations against the time a characteristic form which remains substantially the same over a considerable range of the ratio of the distances of the planets from the Sun.

The transformation is used to eliminate the unknown changes in the eccentricity and longitude of perihelion of the disturbed planet caused by the insertion of a new perturbation. The transformed observations will, in certain cases, show whether a given set of residuals can be accounted for by

the existence of an unknown planet or not. It is then seen that, if the hypothesis appears to be valid, a first approximation to the place of the unknown planet can sometimes be immediately obtained.

The criterion will be applied to the prediction of Neptune by Adams and Leverrier; it shows why the prediction was valid in spite of the large errors in the elements of the orbits given by them.

When the criterion is applied to the modern residuals in the longitude of Uranus, it shows that these residuals cannot be explained on the hypothesis of the existence of an unknown planet with any probable mass or at any probable distance and, in particular, by the action of Pluto. Finally, it gives a clue to the probable source of these residuals, namely, an error in the theory of Uranus.

From these results it easily follows that there is no material in the residuals of the longitude of Uranus which has any value whatsoever for the prediction of an unknown planet, and therefore that the discovery of Pluto within 6° of one of the two places indicated by Lowell was purely accidental, and that all other predictions from this material have the same character. Finally, an upper limit, about one-half that of the Earth, to the mass of Pluto is indicated.

A test of the criterion is furnished by the calculations of Nicholson and Mayall giving the perturbations of Neptune produced by Pluto. It is then shown that existing observations of Neptune are not sufficient to predict the existence of an exterior planet whatever the mass of Pluto may be, and that the early observations of both Neptune and Uranus cannot be satisfied. The later observations of Neptune are consistent with the limit of mass derived from those of Uranus.

2. Since the formula (1.1) is exact, $\Delta x$ may have any value whatever: that used below is $\Delta x = 60°$, giving

$$\sin(x+60°)+\sin(x-60°)-\sin x=0.$$

This has certain properties in connection with perturbation formulæ which are obvious but which, on account of the applications to be made, must be mentioned in detail.

Suppose that we have a set of observations at times $t=0, 1, 2, \ldots$, and that we desire to represent them by the formula

$$f(t)=p+qt+\Sigma C\sin(at+b)+\epsilon_t,$$

where $p$, $q$, $C$, $b$ are unknown, but where one or more of the frequencies $a$ is known; $\epsilon_t$ represents the "accidental" error of the observation. If $n$ be any integer, we have

$$\begin{aligned}\phi(t) &\equiv f(t+n)+f(t-n)-f(t)\\ &=p+qt+\Sigma(1-4\sin^2\tfrac{1}{2}an)C\sin(at+b)+\epsilon_{t+n}+\epsilon_{t-n}-\epsilon_t.\end{aligned}$$

It is evident that the transformation can be as easily applied to the formula as to the observations: it is, in fact, merely a special grouping of the equations of condition.

Suppose that one of the frequencies, denoted by $a_0$, is known and that we are able to choose the integer $n$ such that $a_0 n = 60°$. We then have

$$1 - 4 \sin^2 \tfrac{1}{2} a_0 n = 0,$$

and the term with frequency $a_0$ will disappear from $\phi(t)$.

The comparison of $f$ with $\phi$ shows the following properties :—

(i) The phases and frequencies, as well as $p$, $q$, are unchanged. The coefficient of a term with frequency $a$ receives the factor

$$F_a \equiv 1 - 4 \sin^2 \frac{a}{a_0} 30°$$

in $\phi$. Hence $F_a$ is negative when $a$ lies between $a_0$ and $5a_0$ and is positive when $a < a_0$ ; its extreme values are $+1$, $-3$. If $a = a_0(1 - p)$, where $p$ is small, we have approximately $F_a = 1 \cdot 8p$—a formula which is useful when the relation $a_0 n = 60°$ is not exactly satisfied. For a term of very long period $F_a$ is nearly unity.

(ii) If $N$ be the total number of observations available, and if $A_0$ be the number of units of time in the period $2\pi/a_0$, there will be $N - \frac{1}{3}A_0$ equations for the determination of the unknowns in $\phi$ ; $\frac{1}{6}A_0$ equations at the beginning of the series and $\frac{1}{6}A_0$ equations at the end will be " lost " by the transformation. This fact has considerable importance in the applications to be made below.

(iii) The transformation is applied to the observations by adding the values at times $\frac{1}{3}A_0$ apart and subtracting the middle values.

(iv) The probable error of a residual in $\phi$ is $1 \cdot 7$ times that in $f$. But in the applications to be made below, the probable error of the principal unknown determined from $\phi$ is not much greater than that which would be obtained from the use of $f$.

(v) The effect of the unknown terms $p + qt$ in $\phi$ is a reference of the observations to a base line of unknown inclination and position. The essential characteristics of the curves to be used below will not be much affected by such a reference.

(vi) Isolated observations outside of the continuous series cannot in general be utilised in $\phi$. If a least squares method is to be used to determine the unknowns from $\phi$, any weighting of the observations must be done *after* the formation of $\phi$.

3. Suppose that certain portions of $f$ are theoretically determined by the solution of the equation

$$\frac{d^2 f}{dt^2} + a_0^2 f = a_0^2 \Sigma C_1 \sin (at + b), \tag{3.1}$$

so that

$$f = C_0 \sin (a_0 t + b_0) + \Sigma C_1 \frac{a_0^2}{a_0^2 - a^2} \sin (at + b),$$

where $C_0$, $b_0$ are arbitrary constants. From this we obtain

$$\phi = \Sigma \left\{ \left( 1 - 4 \sin^2 \frac{a}{a_0} 30° \right) \div \left( 1 - \frac{a^2}{a_0^2} \right) \right\} C_1 \sin (at + b).$$

When $a \to a_0$, the limit of the factor of $C_1$ in $\phi$ is $\pi \cdot 3^{\frac{1}{2}}/6 = 0{\cdot}90$. This factor is $0{\cdot}90(1 - \frac{1}{3}k)$ when $a = a_0(1+k)$ and $k$ is small. Hence small divisors due to the integration are not present in $\phi$ and there are no coefficients in $\phi$ which are large compared with those of the corresponding terms in the right-hand member of the differential equation (3.1).

In the application of this result to perturbation theory, it is to be noticed that (3.1) is substantially that which determines the variations of the radius vector, and that it is necessary to find these variations before those of the longitude can be obtained. Hence all large coefficients arising from its integration are avoided. If the method of the variation of the elements be used for the calculation of the perturbations, the transformation eliminates the small divisors which arise through the integration of the long period terms in the equations for the eccentricity and longitude of perihelion.

### *The Transformed Perturbations of the Longitude*

4. The problem is that of the motion of a planet round the Sun disturbed by another planet whose orbit is exterior to that of the disturbed planet. It will first be reduced to its simplest form by the neglect of all perturbations of the co-ordinates which contain either eccentricity or the mutual inclination as factors.

It is well known that the perturbation of the longitude of the disturbed planet then has the form

$$\delta v = m(-A_1 \sin D + A_2 \sin 2D + A_3 \sin 3D + \cdots).$$

In this formula, $D$ is the angle between the radii joining the planets to the Sun and is usually expressed in the form

$$D = nt + \epsilon - n't - \epsilon',$$

where the accented letters refer to the disturbing planet ; $m$ is the ratio of the mass of the latter to that of the Sun expressed in seconds of arc by multiplication by 206265, and the coefficients $A_i$ are functions of the ratio $\alpha$ of the mean distances. An examination of the formulæ which determine them shows that they are all positive so long as $n > 2n'$ ; it is this fact which gives to the transformed equation its characteristic form. For the purposes of this paper we can stop at $A_3$ except in the case of $\alpha > 0{\cdot}6$, and since we are concerned mainly with the *shape* of $\delta v$ and of its transformed value plotted as a function of $t$, attention is directed rather to the ratios of $A_1$, $A_2$ . . . than to their actual values.

The calculation of the values of these coefficients for different values of the ratio of the mean distances is easily carried out by formulæ to be found in the *Mécanique Céleste* of Tisserand (**1**, 365). A selection is given in Table I below.* The values for $\alpha = 0{\cdot}427$, $0{\cdot}475$, $0{\cdot}638$ were calculated from these formulæ, those of the first two being given by Lowell on p. 38 of his *Memoir* : I have tested his coefficients. The values for $\alpha = 0{\cdot}545$ were

* The numbers given in the table may in some cases be in error by some units in the last place given.

extracted with sufficient accuracy from Hill's theory of the action of Saturn on Jupiter, and those for $a = 0{\cdot}723$ from Newcomb's theory of the action of the Earth on Venus. In the last column is shown the approximate application. In these results a value of the disturbing mass 1/50,000 that of the Sun is used.

TABLE I

*Values of the Coefficients* $A_i$

| $a$ | $n'/n$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Approximate Application |
|---|---|---|---|---|---|---|
| | | " | " | " | " | |
| 0·427 | 0·279 | 1·92 | 1·92 | 0·20 | | |
| 0·475 | 0·328 | 2·94 | 3·90 | 0·40 | ... | Pluto on Uranus. |
| 0·545 | 0·402 | 5·59 | 13·6 | 1·13 | ... | Saturn on Jupiter. |
| 0·638 | 0·510 | 17·0 | −670·0 | 10·9 | 2·4 | Neptune on Uranus. |
| 0·723 | 0·613 | 15·7 | − 63·2 | 47·0 | 6·1 | Earth on Venus. |

The large value of $A_2$ in the fourth line is due to the near coincidence of the period of $2D$ with that of the disturbed planet.*

The shapes of the first four curves—$\delta v$ plotted as a function of $t$ over the period of $D$—are shown in fig. 1, with conjunction ($D = 0$) in each case as the middle point. The fourth is substantially a simple sine curve with period that of $2D$. The scale in each case is based on unity for the coefficient of $A_2$.

5. From time to time statements have been published in astronomical journals and elsewhere that the maximum observable effect of one planet on another will take place at or close to conjunction. How completely erroneous such statements are can be seen either from the formula or from the curves in fig. 1. As $D = 0$ at conjunction and $D = \pi$ at opposition, the perturbations *vanish* at those places. Not only is the maximum effect not near conjunction, but in the first three curves it is nearer opposition than conjunction. This is due to the fact that the coefficient of sin $D$ is negative, while $A_2$, $A_3$ are positive. There is a secondary maximum some 40° or so from conjunction, but as long as $n > 2n'$ its height is less than that of the former maximum. In the fourth curve, where $n$ is a little less than $2n'$, so that $A_2$ is negative, the heights of the two maxima are nearly the same, with that approximately 45° from conjunction slightly greater than the other.†

The error referred to is probably due to confusion between the actual curve of motion—which we do not observe—and that which gives the variation of the longitude in terms of the time which is part of what we do observe, the other part being due to the correction of the elements of the disturbed planet to be considered later. The fact is that the perturbations have to be

* The solution does not exist when the coincidence is exact.

† In this connection, the statement of Tisserand (*Méc. Cél.*, **I**, 385) as to the action of Neptune on Uranus is quite erroneous if it is intended to refer to the longitude.

determined from the solution of two differential equations, each of the second order, and these involve the forces in much too complicated a manner for

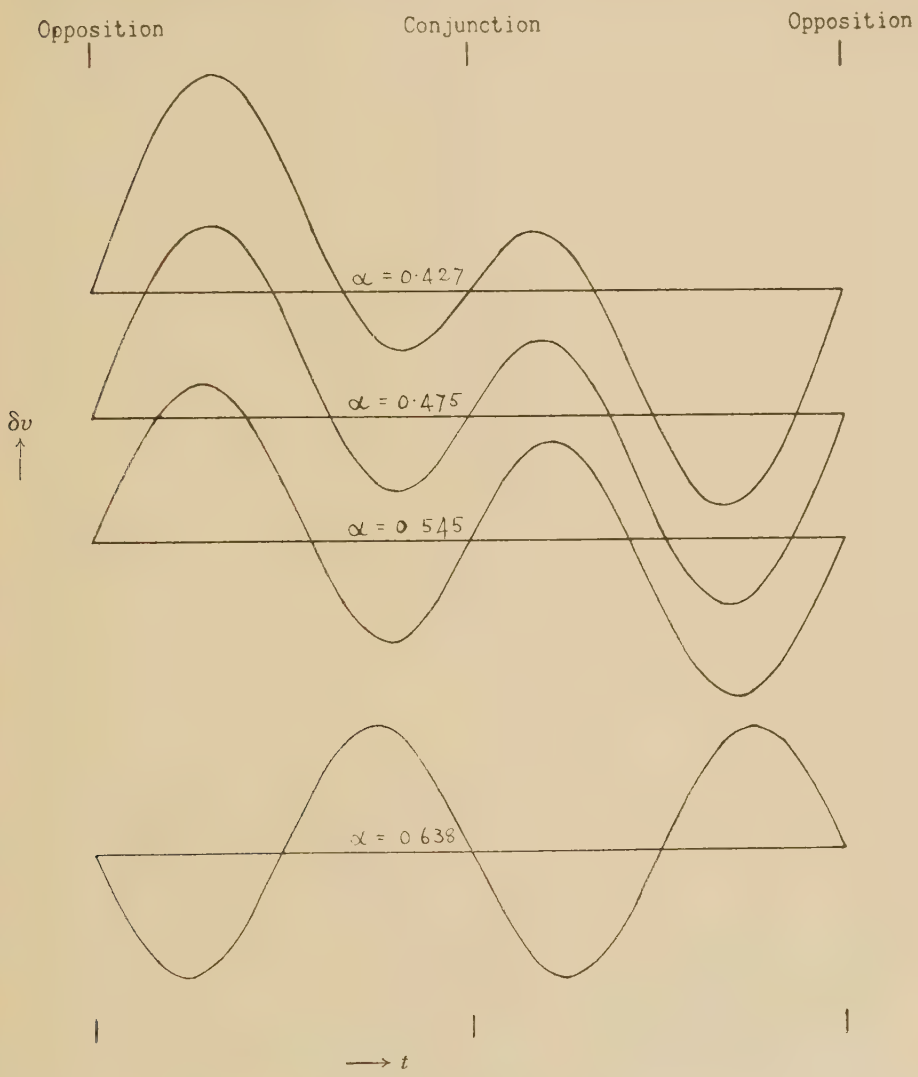



Fig. 1.—Perturbation curves.

safe reasoning by any methods other than those which follow the usual analytical processes, either literal or numerical.

6. Let us now apply the transformation of § 2 with the mean motion of the disturbed planet inserted for $a_0$. The factors $F_a = 1 - 4 \sin^2 (a/a_0)30^\circ$ for the four terms are shown in Table II.

[Table

TABLE II

*Transforming Factors*

| $a$ | $D$ | $2D$ | $3D$ | $4D$ |
|---|---|---|---|---|
| 0·427 | +0·45 | −0·88 | −2·3 | |
| 0·475 | +0·52 | −0·68 | −2·0 | |
| 0·545 | +0·62 | −0·38 | −1·62 | |
| 0·638 | +0·74 | +0·035 | −0·97 | −1·93 |
| 0·723 | +0·84 | +0·38 | −0·29 | −1·08 |

The transformed $\delta v$ denoted by $\phi(v)$ is given by

$$\phi(v) = -m(B_1 \sin D + B_2 \sin 2D + \cdots)$$

where the ratios of the $B_i$ are shown in Table III.

TABLE III

*Ratios of* $B_i/B_2$

| $a$ | $B_1/B_2$ | $B_2/B_2$ | $B_3/B_2$ | $B_4/B_2$ | Position of Maximum | Ratio of Maxima |
|---|---|---|---|---|---|---|
| | | | | | ° | |
| 0·427 | 0·51 | 1·00 | 0·26 | ... | 44 | 3·3 |
| 0·475 | 0·54 | 1·00 | 0·31 | ... | 42 | 4·0 |
| 0·545 | 0·60 | 1·00 | 0·34 | ... | 38 | 5·3 |
| 0·638 | 0·53 | 1·00 | 0·45 | 0·20 | 35 | 5·6 |
| 0·723 | 0·55 | 1·00 | 0·57 | 0·42 | 31 | 5·2 |

The sixth column shows the angular distance of the principal transformed maximum from conjunction, and the seventh column the ratio of the heights of the two maxima. It appears that, for the smaller values of $a$, the angular distance of the principal transformed maximum from conjunction is in the neighbourhood of 40° (with the synodic period represented by 360°), dropping down as $a$ increases, and that in the last three cases the height of the secondary maximum is less than 1/5 that of the primary.

Fig. 2 is drawn to represent $\phi(v)$ for $a = 0{\cdot}475$, $a = 0{\cdot}723$ ; it is sufficient to show the approximate shapes of the curves for the other three values.

*The transformation therefore produces a curve which is characteristic of the perturbations over a considerable range of values of* $a$. It will be seen in § 8 that *the same transformation eliminates the corrections to the eccentricity and longitude of perihelion of the disturbed planet.* We have left only a correction to the base line (corrections to the epoch and mean motion) ; and reference to this latter will not alter the general shape of the curve. *It thus becomes characteristic of a perturbation effect in longitude, and can be used for a test as to whether a given set of residuals can be due to the action of an exterior planet.*

Another possible use of the transformed curve is the construction of approximate general theories covering considerable ranges of distance.

The fact that the simple transformation would give results of this general

character could be, and was, predicted from the analytical formulæ. But it was not foreseen that the position of the maximum and the general shape of the curve would vary so little over precisely the range of distance required in the applications to the discovery of planets from the residuals of Uranus.

7. We have next to see what effect is produced in $\phi(v)$ by the inclusion of the terms which have the eccentricity of the disturbed planet as a factor. A brief examination of known numerical theories shows that in general

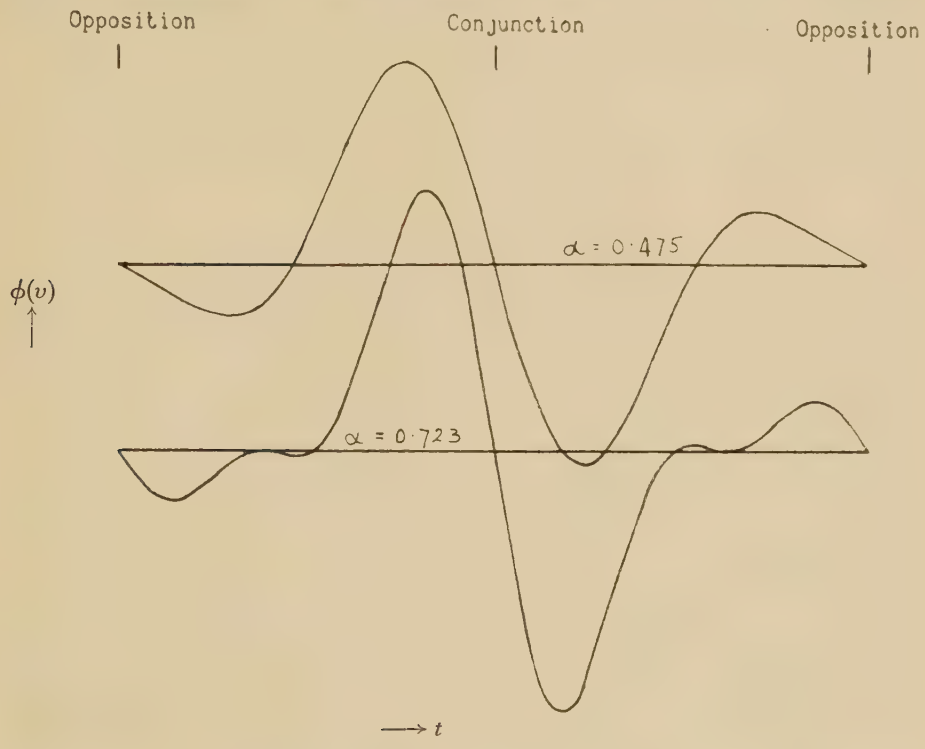



FIG. 2.—Transformed perturbation curves.

there will be at least one term which has a coefficient comparable in magnitude with the coefficients of the terms previously considered, even with an eccentricity as small as 0·05. In order to obtain significant results it is necessary first to limit the scope of the investigation by considering the effect during an interval of time not greater than that of a synodic revolution of the two planets. As the eccentricity of Uranus is 0·047, it will be sufficient to consider magnitudes with this eccentricity.

First, for terms with periods considerably longer than that of the synodic revolution. The curvature produced by such terms is small compared with that in the neighbourhood of the maximum obtained in the transformed longitude, and the only possibility is a magnitude of their coefficients which would seriously alter the maximum. But a long-period term is, over a short interval, expressible approximately by $\alpha+\beta t$, and this merely amounts to an alteration of the unknown base line.

Second, for terms with periods near that of the disturbed planet. Such terms, as we have seen, receive small factors in the transformed longitude.

Third, for other terms with periods of the order of that of the synodic revolution or less. I have not found any general rule and have to rely on calculated results which can be transformed. In all the cases examined, such terms have only a small influence on the results of § 7.

For example let us take the case $a = 0{\cdot}475$, corresponding to the case $a' = 40{\cdot}5$ calculated by Lowell.* He gives as the two principal terms depending on the first power of the eccentricity of Uranus, with $m = 1/50{,}000$,

$$+2''{\cdot}2 \sin (2D - g) - 10''{\cdot}5 \sin (3D - g),$$

where $g$ is the mean anomaly of Uranus. The transforming factors for these are $+0{\cdot}87$, $-0{\cdot}04$, respectively, so that in $\phi(v)$ they become

$$+1''{\cdot}9 \sin (2D - g) + 0''{\cdot}4 \sin (3D - g).$$

These have to be compared with the principal perturbation, $3''{\cdot}9 \sin 2D$. The former has a period nearly twice as long as that of the synodic revolution, and the latter coefficient is small. Thus neither term will greatly change the angular distance between the transformed maximum perturbation and conjunction, nor the ratio between the heights of the two maxima. For terms depending on the higher powers of the eccentricity the effects on these two phenomena are much smaller. As similar results are found for the other cases which I have examined, they are probably true over a considerable range of values of $a$. In particular, they appear to hold for all the values of $a$ considered above.

Similar results are also obtained for the terms dependent on the eccentricity of the disturbing planet, but the coefficients in the transformed longitude are much smaller. This result is significant because it shows that when the eccentricity of the disturbing planet is to be found from the residuals of the disturbed planet over an interval of time less than the synodic period, the " dependence " is small, so that the determination has little significance. The value can, in fact, only be well determined by accurate observations over a longer interval. A numerical test for $a = 0{\cdot}76$ is given in § 22, covering all perturbations.

I conclude that for an approximate test of the validity of a prediction of the existence of an unknown outer planet from the residuals in the longitude of the known planet, formulæ with the limitations of § 6 are sufficient.

### *The Observable Effect of a Disturbing Planet*

8. If $P$ be the sum of the perturbations in longitude produced by any small disturbing force other than those forces which have been taken into account, the observed change of longitude is approximately

$$\delta v = \delta\epsilon + t \,.\, \delta n + 2 \sin g \,.\, \delta e - 2 \cos g \,.\, e\delta\varpi + P,$$

* *Trans. Lowell Obs.*, **1**, 39, 1915.

where $\delta\epsilon$, $\delta n$, $\delta e$, $\delta\varpi$ are changes in the elliptic elements of the disturbed planet and $g$ is its mean anomaly. The insertion of such unknown changes is necessary since the elements have to be found from a comparison of theory with observation, and a change in the theory necessarily demands a change in the quantities determined from the comparison. If the elements have been found from observations over an interval which is many times the period of revolution of the planet, the changes in the elements caused by the presence of $P$ are usually small compared with the magnitude of $P$. But if the interval be one revolution or less, the changes may be of the same order as $P$.

When the transformation of § 2 is applied to $\delta v$ with $2\pi/a_0$ as the period of revolution, that is, of $g$, the coefficients of $\sin g$, $\cos g$ disappear and we have

$$\phi(v) = \delta\epsilon + t \,.\, \delta n + Q,$$

where $Q$ is the sum of the transformed perturbations.

When $Q$ is known we have a set of equations of condition to find the two unknowns $\delta\epsilon$, $\delta n$ instead of the four present in $\delta v$. After these have been found we might return to the original equations to find $\delta e$, $\delta\varpi$.

But the chief object in view here is not the determination of these unknowns but a test as to whether the observed residuals can be satisfied by $P$ with suitable values for $\delta\epsilon$, $\delta n$, $\delta e$, $\delta\varpi$, and it has been shown that this is equivalent to a test as to whether the transformed residuals can be satisfied by $Q$ with suitable values for $\delta\epsilon$, $\delta n$ only. It is equivalent in the sense that if the latter condition is not fulfilled neither is the former ; but if the latter is fulfilled, the former may or may not be fulfilled, since the latter does not necessarily contain the full effect of the equations of condition.

As the test is to be applied only over an interval not greater than the synodic revolution where $P$ is the result of the disturbance caused by an outer planet, we may limit the terms to be contained in $P$. We have seen in § 7 that terms of long period may be approximately represented in such an interval by $\alpha + \beta t$, and therefore may be supposed to be included in the unknown $\delta\epsilon$, $\delta n$, and the magnitudes of the remaining terms have been discussed in §§ 6, 7.

A rough preliminary test can be made by comparing a graph of the transformed residuals $\phi(v)$ with that of the transformed perturbations $Q$, the effect of the terms $\delta\epsilon + t\delta n$ being merely the reference of either graph to a base line inclined to that originally used. If the comparison appears to satisfy the test, we can go further and see what information can be obtained by the comparison. If it is not satisfied, the hypothesis must be abandoned.

## *The Discovery of Neptune*

9. Let us apply the results to the problem of Leverrier and Adams, the solution of which led to the discovery of Neptune. The observational material available to them is shown by the black dots in fig. 3 ; they represent the residuals in the heliocentric longitude of Uranus after the

effects of Jupiter and Saturn had been included in the theory of the motion of the planet, and after the elements of Uranus had been determined by the comparison of this theory with observation. Actually, the residuals used by Adams are shown, but the original material used by Leverrier is not

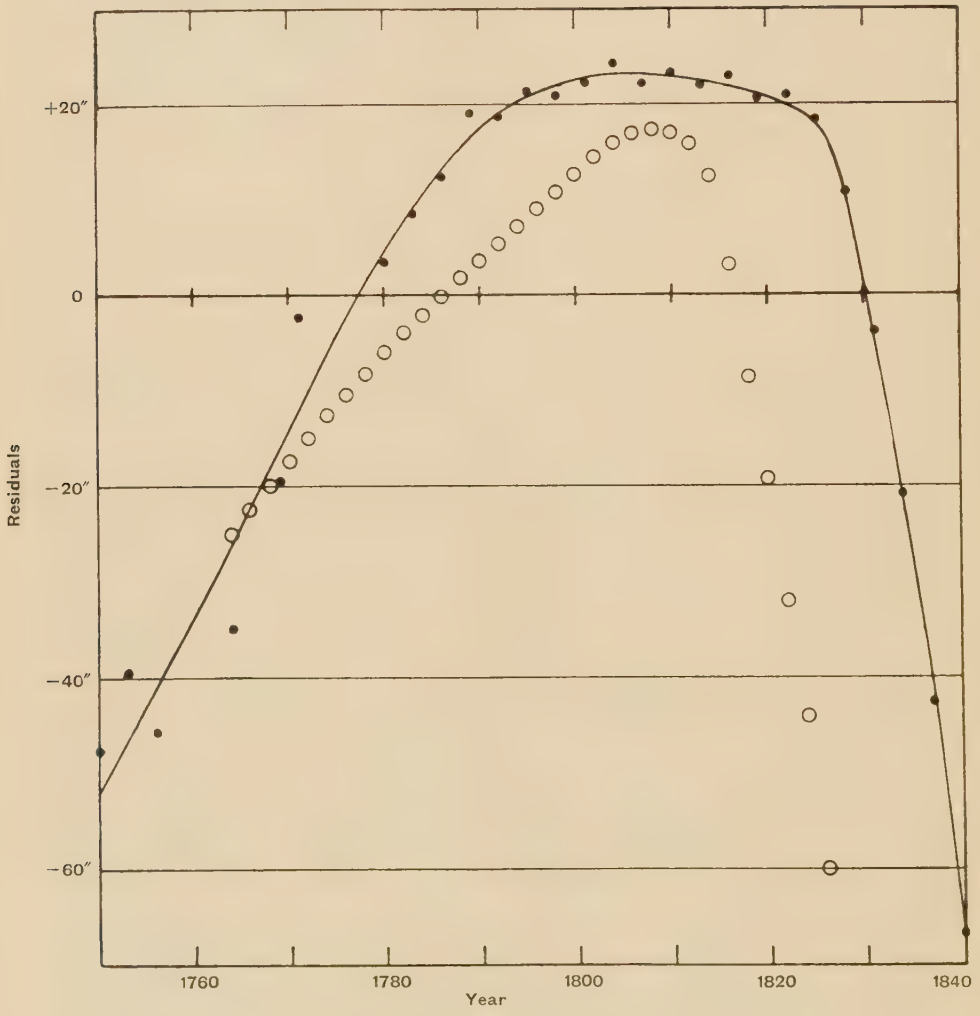



Fig. 3.—Residuals used by Adams are shown by dots ; smoothed residuals by the curve ; transformed residuals by small circles.

substantially different. To the residuals shown in the figure must be added three groups of residuals :

$$1690,\ +61''\ ;\qquad 1712,\ +93''\ ;\qquad 1715,\ +74''.$$

Since the errors of observation of the material in the figure are obviously small compared with the systematic changes, I have drawn a smooth curve to represent the approximate residuals from 1750 to 1840, where the series ends.

The open circles in the figure show the transformed residuals deduced from the smooth curve. Since the period of revolution of Uranus is 84 years, the transformation is made by adding ordinates 28 years apart and subtracting the middle ordinate, the result being plotted under the date of the latter. All that can be deduced from the three additional groups is a probable large positive residual about 1704.

Comparison of the curve formed by the open circles with the curve of fig. 2 shows that they *may* represent the same phenomenon. The statement at this stage cannot be made more definite because the transformed residuals cover an interval of only 62 years, while the synodic period of Uranus and the hypothetical planet is probably at least twice this interval. Assuming, however, that the hypothesis is valid, we can immediately obtain certain information about the unknown planet.

In the first place, the maximum of the transformed residuals must evidently be that of the transformed perturbation curve shown in fig. 3. That any probable change in the base line will not alter the date of this maximum by more than three years is evident with the aid of the transformed group result near 1704. This date is probably within two years of 1810.

Next, a reference to Table III shows that the position of this maximum is in the neighbourhood of 40° from conjunction—that is, one-ninth of the synodic revolution. With the distances assumed by Adams and Leverrier the synodic period is about 135 years, corresponding approximately to a value between the second and third lines of the table : this gives an interval of (40/360) . 135 years or 15 years between the maximum and conjunction. The actual synodic period of Neptune and Uranus is 171 years, which gives (35/360) . 171 years, or 19 years between the maximum and conjunction. With 1810 as the date of the maximum these give 1825 and 1829, respectively, as the date of conjunction.

This approximation is, however, rough because it is made by neglecting the terms produced by the eccentricity of Uranus. The point is, however, at the moment, not the actual date of conjunction but the variation in this date due to the assumption of different distances, and we have seen in § 7 that this variation will probably be much smaller in the transformed perturbation than the actual error of the date. In other words, a correct calculation with an erroneous distance will cause only a small error in the date of conjunction.

10. To go further, some numerical calculation is necessary. As a test, I took the terms given by Adams with arguments $D$, $2D$, $3D$, $2D-g$, which give the largest effects in the transformed function, so that $Q$ was expressed by

$$Q = -mB_2\left\{\frac{B_1}{B_2}\sin D + \sin 2D + \frac{B_3}{B_2}\sin 3D + \frac{B'}{B_2}\sin(2D-g)\right\}.$$

An earlier estimate of the date of conjunction had given a date later than 1830. As $g=0$ about 1840, the calculation was simplified by taking this latter as the date of conjunction to be corrected by calculation. Now a

variation in the date of conjunction is equivalent to a variation of $\epsilon'$. The corresponding change in $Q$ is given by

$$\delta Q = mB_2 \left\{ \frac{B_1}{B_2} \cos D + 2 \cos 2D + 3\frac{B_3}{B_2} \cos 3D + \frac{B'}{2B_2} \cos (2D - g) \right\} \delta\epsilon'.$$

With the four unknowns $\delta\epsilon$, $\delta n$, $mB_2$, $mB_2\delta\epsilon'$, the equations of condition at the dates 1777, 1791, 1812, 1826 were solved, that at 1805 being calculated as a test. The result actually gave a date for conjunction close to 1840, and it was evident that the transformed residuals between 1764 and 1826 would be satisfied within a few seconds of arc.

That this date would be close to the actual date was due to the fact that the variations of the coefficients $B_1/B_2$, $B_3/B_2$, $B'/B_2$ for different distances would not be great. Most of the change is taken up by the unknown $mB_2$. It is due to this fact that a nearly correct date of conjunction was deduced from an erroneous mean distance.

11. What about the eccentricity and longitude of perihelion of the unknown planet? The values deduced by Leverrier and Adams bear no resemblance to the corresponding elements of Neptune. In § 7 I have pointed out that the terms in the transformed perturbation which have the eccentricity of the disturbing planet as a factor appear to have either long periods or small coefficients. In neither case will they much affect the determination of the date of conjunction when the latter is found from observations over an interval of time much less than the synodic period : as we have seen, the long period terms are mainly absorbed in the changes $\delta\epsilon$, $\delta n$. Thus the use of these was mainly to satisfy better the earlier observations : the erroneous distance naturally produced large errors in the determination of these quantities.

That this is the case can also be seen from an examination of Adams's work. In a second attempt he tried the use of a slightly different distance, namely, 3 per cent. smaller, and found large changes in the calculated eccentricity and longitude of perihelion of the unknown planet, and large changes in the corrections to the elements of Uranus, but only a small change in the predicted date of conjunction.

Finally, the prediction of the date of conjunction being obtained as near 1840, the prediction of the place of the planet in 1846, when it was first detected, could not vary much. The actual motion of Neptune in six years with a period of 165 years is $13^\circ{\cdot}1$, with a period of 230 years it would be $9^\circ{\cdot}7$, a difference of less than $3^\circ{\cdot}5$. Actually the planet was found within $2^\circ{\cdot}5$ of the predicted places.

12. It may be objected that the method I have adopted is not the same as the methods of Leverrier and Adams. The answer is that, since my method is merely a particular grouping of the equations of condition which enables me to obtain a first approximation to the place of the planet with but little calculation, it is necessarily equivalent to theirs. It should not, however, give final results with the same accuracy for most of the unknowns, the possible exception being the date of conjunction.

13. One may ask whether it is possible to obtain an approximation to the

distance or, what is the same thing, to the period of the unknown planet. Both Adams and Leverrier guessed at the distance. The foregoing analysis shows that the principal unknown magnitude which contains it is $mA_2$ or $mB_2$. Now $A_2$ has a factor which is, in effect, the cube of the ratio of the mean distances. Thus, though $mA_2$ can be found, it is doubtful if we can separate its two factors : an error in $A_2$ will be almost entirely compensated for by an error in $m$.

In Leverrier's prediction the product is within 10 per cent. of the corresponding product for Neptune ; in that of Adams it is erroneous by some 40 per cent. The difference is partly due to the fact that Leverrier's assumed distance is nearer to the distance of Neptune than that of Adams, and possibly in part to a method of procedure of the latter which is open to slight criticism and which is discussed in § 14 below.

If this statement be compared with the fact noted in § 5, namely, that the actual maximum effect of the perturbation in longitude is some 120° to 130° from conjunction, it appears paradoxical. The fact is that, in this connection, rates of change are more important than actual changes, as shown in the use of the transformed equations : by the device adopted in this paper, rates of change have, in fact, been brought out without their usual defect, namely, a great increase in the influence of errors of observation.

14. In the description of his treatment of the perturbations Adams makes the following statement: " It is easily seen that the series expressing the correction of the *mean* longitude in terms of the corrections applied to the elements of the orbits is more convergent than that which gives the correction of the *true* longitude, and the same thing is true for perturbations of the mean longitude as compared with those of the true. The corrections found above [observed corrections] were accordingly converted into corrections of mean longitude by multiplying each of them by the factor $r^2/ab$, $r$ being the radius vector, and $a$ and $b$ being the semi-axes of the orbit." The meaning of this process is not clear. If it is applied to both observation and theory, there is no objection to it with one exception, namely, that the corrections to the elements of Uranus, which are contained in the observations, are not the same as before. In particular the correction to the mean motion is now given by $t\delta n(r^2/ab)$, instead of by $t\delta n$, and this change does not appear to have been made. The numerical effect is not very large, but it may have been sufficient to account for the greater deviation of the predicted place than that of Leverrier, particularly as the last residual, which is important, is changed from $-66''{\cdot}64$ to $-73''{\cdot}09$. The change may have been taken care of in some other way : Adams does not always give full details. My object here is not a criticism of Adams's work, but merely an attempt to explain a difficulty which others also have found in trying to understand his paper.

### *Predictions from the Modern Residuals of Uranus*

15. After Neptune had been discovered, the elements of its orbit found, and its actual effect on the motion of Uranus computed, it was evident that

the residuals of the heliocentric longitude of the latter were very small. Gaillot, using Leverrier's theory, corrected its elements and published a set of residuals * which will be used here, partly because they appear to be homogeneous, and partly because they have been used as the basis for most of the attempts at the prediction of another planet.

The group residuals from 1780, as given by Gaillot, are plotted in fig. 4. Not included in the figure are the following :—

$$\left.\begin{array}{llll} 1690\text{–}1715, & +2''\cdot14 \pm 1''\cdot32\,; & 1764\text{–}1768, & -0''\cdot58 \\ 1750\text{–}1756, & +4\cdot45 \pm 1\cdot24\,; & 1769\text{–}1771, & +3\cdot78 \end{array}\right\} \pm 1''\cdot26.$$

The probable errors are those assigned, but not used, by Lowell. The first group contains one observation in 1690, one in 1712 and three in 1715, which last differ amongst themselves by $6''\cdot5$. In the second group we have two in 1750 and one in 1756. The third and fourth groups contain observations, eight of which were made within a month and differed from one another by as much as $16''\cdot9$. Most of the groups in the interval 1780–1830 contain two or three observations ; after 1830, the number generally runs up to ten or more.

The scale of fig. 4 should be noted. Nearly all the residuals in it are less than $2''$, while those dealt with by Adams ran up to over $60''$. There is not much reliable evidence of systematic change : what there is forms the sole basis for the prediction of an unknown planet. It must be remembered in this connection that sensible oscillations with periods of less than thirty years cannot be caused by the action of an outer planet : and terms with very short periods have always very small coefficients.

A general idea of the errors of observation can be obtained from the figure. Under no hypothesis of perturbative effects by an outer planet can rapid changes like those, for example, near 1830, 1836, be explained. It is known from several sources that before 1840 observational errors in longitude were apt to be large ; those of the Sun, for example, ran up to $4''$. Practically the only significance which can be attached to the residuals between 1750 and 1840 is their approach to a straight line. After 1850 there appears to be an oscillation with a maximum about 1870, a minimum near 1890 and a later maximum.

16. The first question to be answered is : Can these oscillations be explained under the hypothesis of the action of an exterior unknown planet? The effect of the transformation of § 2 on the observations from 1790 is shown in fig. 5. This is to be compared with the characteristic transformed perturbation curve of fig. 2.

The presence in the transformed curve of the *two* minima of nearly equal depth near 1830 and 1870, and of the *two* maxima near 1855, 1885, shows that the answer is definitely in the negative. If we reject these oscillations as due to some other cause, there is nothing left but a straight line from 1780 to 1903. If a planet does not produce sensible oscillations it is of course impossible to do anything in the way of prediction.

* *Ann. de l'Obs. de Paris (Mém.)*, **28**, A77, 1910.

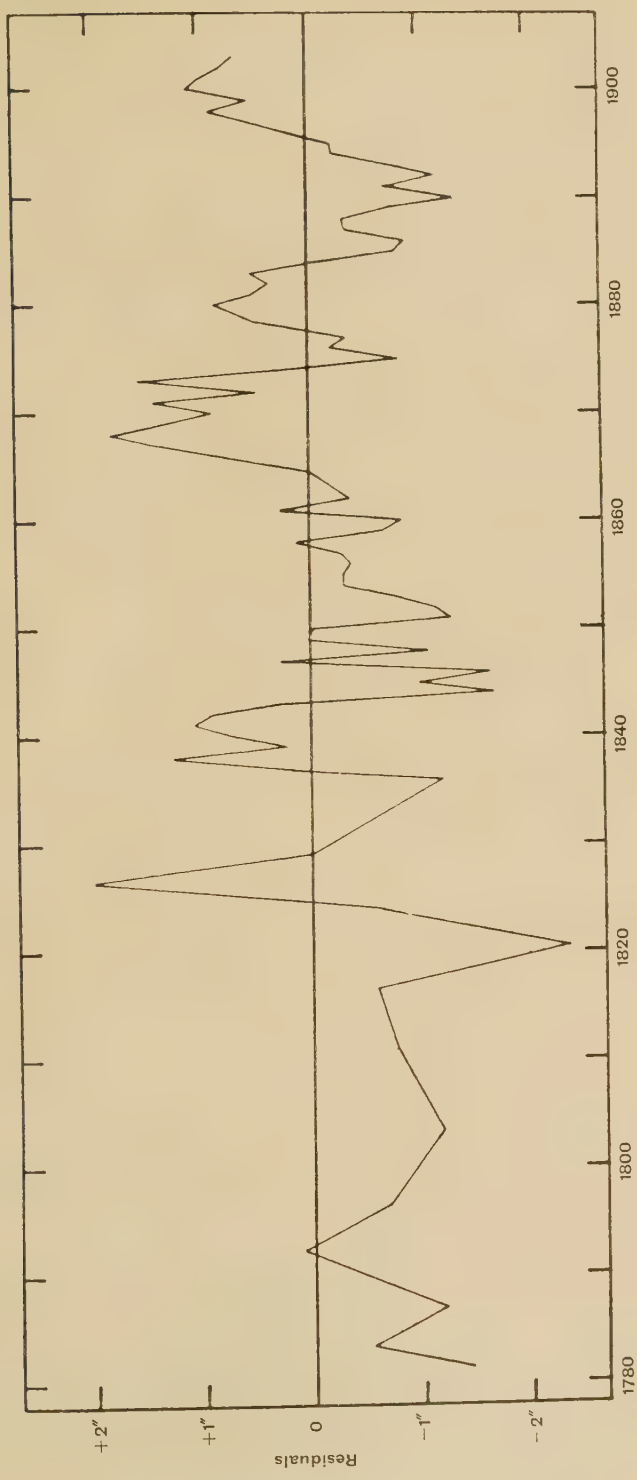



FIG. 4.—Gaillot's residuals of Uranus.

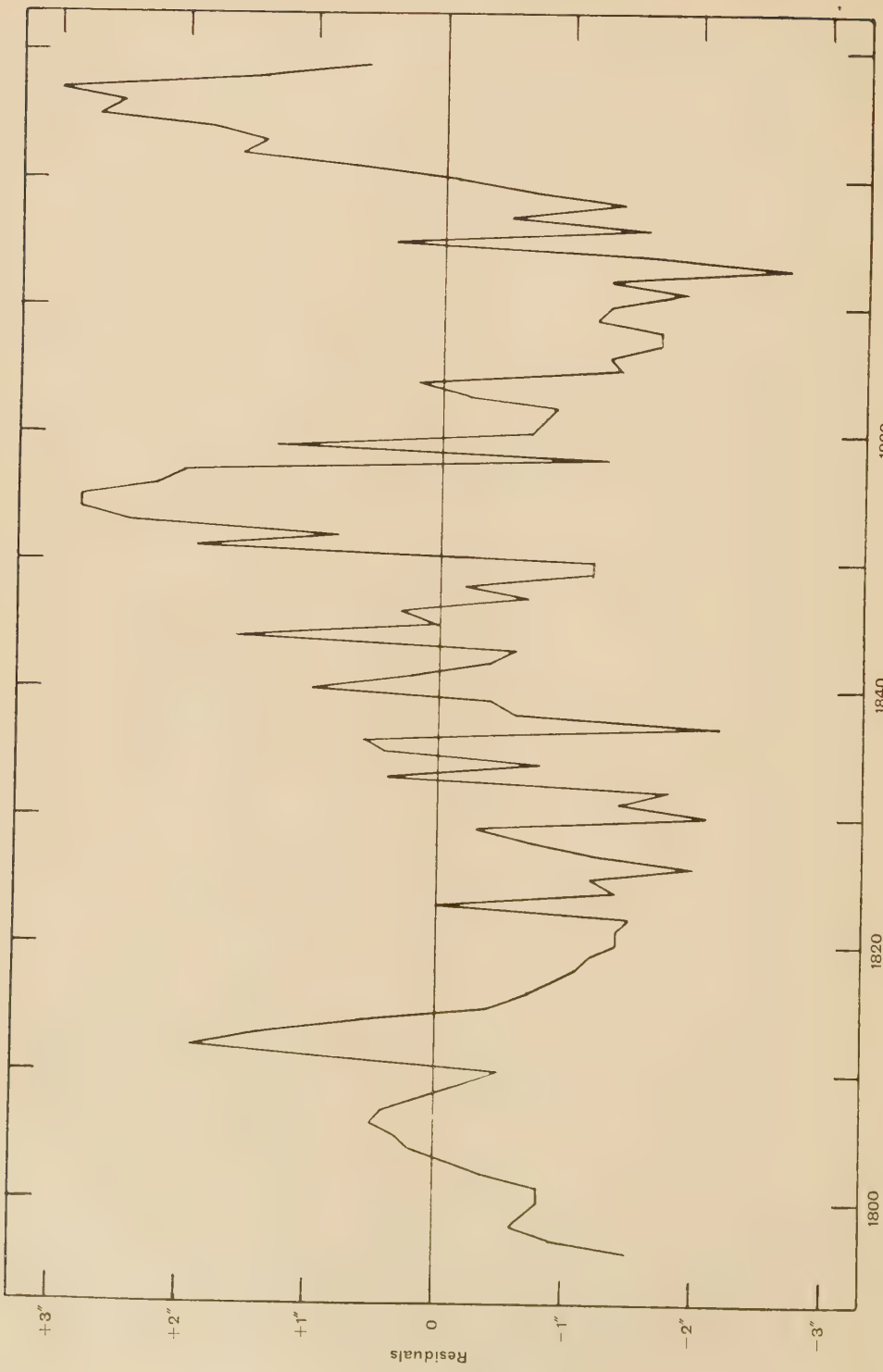



FIG. 5.—Gaillot's residuals transformed.

17. I have dealt with certain phases of Lowell's prediction elsewhere.* It has been shown that it was, in effect, a result of so adjusting the elements of Uranus that the perturbations produced by the unknown planet were relatively small during the interval of continuous observation, and that the predictions of conjunction or opposition and of perihelion were entirely due to this attempt. It is shown in § 16 that these small deviations bear no resemblance to the actual residuals of Uranus.

Some further remarks may be added. At the date of publication of my earlier paper, the elements of the orbit of Pluto had not been well determined. Those available while the paper was in proof, namely, those published in a circular of the Lowell Observatory dated 1930 May 1, contained an eccentricity of 0·91 which, of course, put the object outside of any possibility of prediction by the methods of the planetary theory. The later orbits gave a much closer approximation to those of Lowell. Nevertheless, not only does the orbit not satisfy the residuals of Uranus after 1750 and, in particular, the later and more accurate data, but if the mass of Pluto is anywhere near that of the Earth or less, it causes alterations in the residuals before that time which are less than their probable errors.

18. The significance of the fact that Lowell gives two orbits which differ only in the fact that the epoch $\epsilon'$ in one is 180° away from that in the other, may be pointed out. According to his results the residuals can be satisfied by two sets of values :

$$\delta v = C_1 + P_e + P_0, \qquad \delta v = C_2 + P_e - P_0,$$

where $P_e$ contains the perturbations having even multiples of $\epsilon'$ in their arguments and $P_0$ those having odd multiples, while $C_1$, $C_2$ are the accompanying changes in the elements of the orbit of Uranus. Evidently, they must equally well be satisfied by

$$\delta v = \tfrac{1}{2}(C_1 + C_2) + P_e.$$

In other words, all the terms containing odd multiples of $\epsilon'$ can be neglected without affecting the prediction. This leaves only one term which will give a sensible oscillation in an interval shorter than the synodic period, namely, $3''{\cdot}9 \sin 2D$, with a mass 7 times that of the Earth. Its effect is considerably changed by changes in the eccentricity and longitude of perihelion of Uranus which give a term with a period of 84 years. Its period is about 55 years, and fig. 5 shows that it cannot satisfy the oscillations in the residuals of Uranus during the nineteenth century.

19. The oscillation shown in the later residuals of Uranus and exhibited in figs. 4, 5 appears to be real, and consequently needs explanation. It has an apparent period of some 40 years and a maximum coefficient of about 1″ : a coefficient with this period would be doubled in the transformed curve. Since the period of revolution of Uranus is 84 years, it suggests an error in the theory of the elliptic term with argument $2g$ and having as factor the square of the eccentricity of Uranus—an error which might be caused by incomplete calculations when changes in the elements of Uranus were made. A more probable explanation is, however, that the long-period

* *Proc. Nat. Acad. Sci.*, **16**, 364–371, 1930.

changes in the eccentricity and longitude of perihelion, which for the perturbations produced by Neptune are quite large, have not been fully applied to the elliptic term with argument $2g$. This is the more probable because there is evidence of a smaller amplitude of the oscillation before 1840, indicating the presence of two or more terms with periods which differ a little from that of $2g$ and which more nearly cancel one another in the earlier period.

If the oscillation be such an error in the theory, its correction leaves nothing to be explained except the early and very doubtful residuals. But in this case, as stated in § 17, there is no material on which to base a prediction.

20. I have dealt only with the predictions made by Lowell because his is the only investigation carried out on recognized lines in which the steps are given in sufficient detail and with sufficient apparent accuracy to permit of interpretation and criticism as to their real significance. Certain statements made by Professor W. H. Pickering, particularly that which refers to the maximum perturbation in relation to conjunction,* raise considerable doubt as to the validity of his method for computing the perturbations. Moreover, the diagram which he gives of the perturbations produced by Pluto, apparently using Lowell's mass, bears little resemblance to the actual perturbations. Lowell's planet approximately satisfied the large residual of the 1753 group, and if we add to the diagram the necessary changes in the orbit of Uranus to achieve the same result, we obtain large deviations in the latter half of the nineteenth century which neither the observed residuals nor the orbit show.

The fact is that graphical methods are convincing only when they can be translated into analysis. In the present case it should be shown that they are equivalent to the solution of the two differential equations, each of the second order, which is necessary to obtain the perturbations of the longitude. It is true that Herschel, in his *Outlines of Astronomy*, devotes much space to a graphical description of perturbative effects, but most of the work is qualitative and refers to changes in the elements produced by the action of a third body, except that on the Moon, where he follows Newton. Adams (*Coll. Works*, **1**, 70, 1896) stated that he had obtained a graphical method for computing perturbations, but gave indications only as to its nature, and there appears to be no evidence that it was ever published.

21. Such indications concerning the mass of Pluto as we can obtain from the observed residuals of Uranus point to a mass considerably less than that of the Earth. A mass equal to that of the Earth would give the principal perturbation the value $0''{\cdot}6 \sin 2D$, and this would have a coefficient of $1''$ in the transformed residuals. As I have shown in § 16, there is no evidence of the existence of such a term. An upper limit less than half the mass of the Earth is indicated.

Any attempt to determine the mass from the known residuals of Uranus cannot lead to results of any value as long as there is doubt as to the accuracy of the theory of the motion of Uranus. What is needed is a knowledge of

* *M.N.*, **91**, foot of p. 814, 1931.

the coefficients of the short period terms due to the action of Neptune to one or two tenths of a second of arc. Even with such knowledge another century of accurate observations appears to be necessary for a determination which shall have a probable error less than a quarter of the Earth's mass.

### *The Action of Pluto on Neptune*

22. *Test of the Criterion.*—The eccentricity of Pluto's orbit (0·25) and the ratio of the mean distance of Neptune to that of Pluto (0·76) are both so large that the theory on which the criterion is based cannot properly be applied. But as Nicholson and Mayall * have calculated by mechanical quadratures the changes in the heliocentric longitude of Neptune produced by the action of Pluto with an assumed mass equal to that of the Earth, it was of some interest to see how closely the main feature of the criterion, namely, the interval between the maximum and conjunction in the transformed perturbations, would agree with that deduced on the basis of a zero eccentricity. This interval, as shown above, is independent of the assumed mass.

Their calculations give the perturbations in longitude at four-year intervals from 1796 to 1928 : these are shown by the dots in fig. 6. Since the period of revolution of Neptune is 164·8 years, and one-third of this is 55 years, sufficient accuracy in the transformed perturbations will be obtained by adding the values 56 years apart and subtracting from the sum the middle value. This process gives results from 1824 to 1900, which are shown by the small circles in fig. 6. The indicated maximum is at 1866, the small correction of the base line (which they also give) throwing this maximum back to 1864. As conjunction occurred in 1892 we have an interval of 28 years which, taken in conjunction with the synodic period of 486 years, gives an angle of 21°. Extrapolation from Table III indicates an angle of about 30° when the eccentricity is neglected.

It is, however, possible to take partial account of the effect of the eccentricity on the results of Table III. The angle 30° is only one-twelfth of the synodic revolution, and during this short interval most of the change in the equation of the centre can be included in the motion of the angle $D$. A brief calculation shows that near 1900 this is equivalent to a diminution of the motion of $D$ by 20 to 30 per cent. This means that the angle of 30° should be reduced to less than 24°—bringing it quite close to the actual angle of 21°. This argument is given with some reserve, since other modifications of the same order of magnitude may be necessary. Nevertheless, it suggests the possibility of development of a theory of the criterion which shall take better account of the actual orbits of the disturbed and disturbing planets.

23. *The Residuals in the Longitude of Neptune.*—These have been deduced by Dr. J. Jackson,† after correction of the orbital elements but without any allowance for the action of Pluto, and are shown in fig. 1 of the paper of

* *Mount Wilson Contribution*, No. 417 ; *Ap. J.*, **73**, 1–12, 1931.

† *M.N.*, **90**, 728–732, 1930.

Nicholson and Mayall. They are given at four-year intervals from 1848 to 1928, with a single group at 1795. Applying to them the same transformation as was applied to the Nicholson and Mayall calculation of the action of Pluto, we obtain seven residuals at four-year intervals from 1876 to 1900. These, after the alteration of the base line referred to above, namely, by the amount $-0''\cdot72 - 0''\cdot18T$, where $T$ is the number of centuries from 1870, are shown by the barred circles in fig. 6. As is to be expected, they oscillate about the transformed perturbation curve, since the constants of

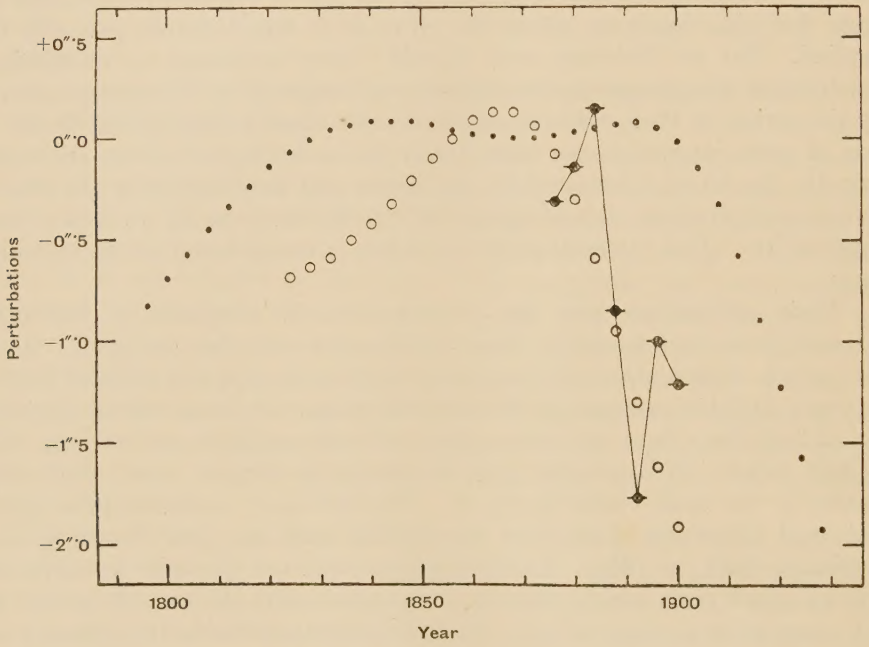



FIG. 6.—Nicholson and Mayall perturbations of Neptune's longitude are represented by dots, the transformed perturbations by small circles, the transformed observations by barred dots.

the orbit of Neptune and the mass of Pluto have been so determined as best to fit the observations.

This transformed material, referred to an arbitrary base line, is evidently quite useless for the purpose of predicting an unknown planet, but it may be questioned whether the full value of the observations is retained in the transformation so far as the prediction is concerned. The answer is that the synodic period of Neptune with that of any probable outer planet will certainly be greater than 300 years, and that the interval of continuous observation is only a little over a quarter of this : such an interval is too short for the purposes of prediction with the very small actual deviations shown. As I have pointed out above, the early residuals play practically no part in predictions at dates just after the end of those of the continuous observations.

24. *The Mass of Pluto derived from Observations of Neptune.*—In their paper, Nicholson and Mayall derive a value $0\cdot94 \pm \cdot25$ that of the Earth when

the early observations of Lalande are included, but they point out that if these observations are rejected, any mass less than that of the Earth satisfies the observations almost equally well. The latter statement implies that a mass equal to that of the Earth is an upper limit.

We have, however, seen that such a mass will not satisfy the early residuals of Uranus, and that the later residuals point to a mass not greater than half that of the Earth. In these circumstances, it is not possible to draw any conclusions concerning the mass from the early residuals of either planet. The upper limit determined from the later residuals of Uranus is within that found from those of Neptune.

Another century at least of observations of Neptune will be needed to obtain a value of the mass of Pluto with a probable error less than a quarter of the Earth's mass. These must be compared with a theory of the perturbations of Neptune by Uranus which shall give the principal short-period coefficients accurately within 0″·2. It is doubtful whether reliance can be placed on Newcomb's theory to this degree of accuracy. Owing to the fact that the period of revolution of Neptune is nearly twice that of Uranus, high accuracy in the theoretical work is difficult of attainment, and certain coefficients are quite sensitive to small changes in the constants of the orbits. We have the additional difficulties caused by the fact that the periods of revolution of Pluto and Neptune are nearly in the ratio 3 : 2. The result of these near integral relations between the periods is that certain of the perturbations of Neptune produced by Uranus will have nearly the same periods as those produced by Pluto. Thus any error in the calculation of the former may be ascribed to the latter, and so produce an erroneous result when comparisons with observation are made in order to determine the mass of Pluto.

25. Just as this paper was completed, a discussion of the orbit and mass of Pluto by E. C. Bower * reached me. His statements concerning the possibilities of determining the mass of Pluto from the residuals of Uranus and Neptune are the same as those reached by the arguments given above, and I have not found it necessary to make any changes as a consequence of the publication of his work. His upper limits of mass ratio to that of the Earth, namely, 0·7 from assumptions of albedo and density with observed magnitude, and 0·3 from observed diameter and assumed density, may be compared with the upper limit 0·5 derived here from the residuals of Uranus. He concludes that the present most probable value for the actual mass is 0·1. A prediction with any such value of the mass, based on the gravitational action of the planet and the available observations, has no value.

* *Lick Obs. Bull.*, No. 437, 1931.

*Yale University :*
1931 *October* 13.

---

*Erratum*

*M.N.*, **91,** 829, second footnote, *for Astrophysical Journal*, **69,** 145, 1929, *read* **69,** 245, 1929.